DEPARTMENT OF MATHEMATICS
BIRLA INSTITUTE OF TECHNOLOGY MESRA, RANCHI
MA103 Mathematics-I, Session: (MO-2023)
Assignment - 4 (Module IV)

---

1. Prove that

   a) $B(m,n) = B(m+1,n) + B(m,n+1)$

   b) $B(n,m) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$

   c) $\int_0^1 y^{q-1} \left(\log \frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}, \quad p > 0 \text{ and } q > 0$

   d) $\int_0^\infty \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, \quad c > 1$

   e) $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

   f) $\Gamma(m)\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m), \quad m > 0$

2. Prove that $\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n+2}{2}\right)}$ and hence find the value of $\Gamma\left(\frac{1}{2}\right)$.

3. Show that

   a) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

   b) $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx = \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{2}\right)}{3\Gamma\left(\frac{4}{3}\right)}$

   c) $\int_0^2 x(8-x^3)^{\frac{1}{3}}\, dx = \frac{16\pi}{9\sqrt{3}}$

   d) $\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\, dx = \sqrt{\frac{\pi}{a}}$

   e) $\int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx = \frac{3}{2}\sqrt{\pi}$

   f) $\int_0^\infty \frac{1}{1+y^4}\, dy = \frac{\pi\sqrt{2}}{4}$

   g) $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^5\theta\, d\theta = \frac{8}{315}$

   h) $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{5\pi}{32}$

4. Evalute the multiple integrals.

   a) $\int_1^2 \int_y^{y^2} dx\, dy$

   b) $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{\frac{y}{\sqrt{x}}} dy\, dx$

   c) $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz\, dy\, dx$

   d) $\int_0^{\frac{\pi}{4}} \int_0^{\ln(\sec v)} \int_{-\infty}^{2t} e^{2x} dx\, dt\, dv$

5. Change the order of integration of the following double integral

   a) $\int_0^a \int_0^x \frac{\psi(y)}{\sqrt{(a-x)(x-y)}} dy\, dx$

   b) $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y) dy\, dx$

6. Change the order of integration of the following double integrals and hence evaluate the same

a) $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx$

b) $\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dxdy$

c) $\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

d) $\int_0^\infty \int_0^x xe^{\frac{-x^2}{y}} dy\, dx$

e) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy\, dx$

f) $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy\, dx$

7. Transform from Cartesian to Polar form and then evaluate

   a) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{0} \frac{4\sqrt{x^2+y^2}}{1+x^2+y^2} dx\, dy$

   b) $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x+y}{x^2+y^2} dy\, dx$

8. Evaluate $\iint \frac{r}{\sqrt{a^2-r^2}} drd\theta$ over the loop of the lemniscate $r^2 = a^2\cos(2\theta)$.

9. Find the area that lies outside the circle $r = 1$ and inside the Cardioid $r = 1 + \cos\theta$.

10. Find the area of the region common to the interiors of the Cardioids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$.

11. Find the volume common to the two cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$.

12. Find the volume bounded by the cylinder $y^2 + 4z^2 = 16$ and the planes $x + y = 4$ and $x = 0$.

13. Find the volume of the portion of the sphere $x^2+y^2+z^2 = a^2$ lying inside the cylinder $x^2+y^2-ax = 0$.

14. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 - 2ay = 0$ and the plane $z = 0$.

15. Find the volume enclosed by the cylinders $x^2 + y^2 - 2ax = 0$ and $z^2 - 2ax = 0$.

16. Evaluate $\int_0^{\frac{2}{3}} \int_y^{2-2y} (x + 2y)e^{-(x-y)} dxdy$ using the transformation $u = x + 2y$ and $v = x - y$.

17. Find by triple integration the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.