MA107-Mathematics-II
Topics-Module-IV

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MODULE-IV

- FUNCTIONS OF COMPLEX VARIABLE
- LIMIT,CONTINUITY,DIFFERENTIABILITY
- ANALYTICITY
- CAUCHY REIMANNS EQUATIONS
- HARMONIC FUNCTIONS
- HARMONIC CONJUGATE
- CAUCHY’S THEOREM
- CAUCHY’S INTEGRAL THEOREM

CONTINUED…

- TAYLOR’S & LAURENT SERIES EXPANSIONS
- SINGULARITIES AND TYPES OF SINGULARITY
- RESIDUES
- RESIDUE THEOREM

1. Complex Variables & Functions

Complex numbers : $£ = \{ z = x + i y \; ; \; \forall \, x, y \in ¡ \; \}$ (Ordered pair of real numbers)

Complex conjugate : $z^* = x - i y$

Polar representation : $z = r\, e^{i\theta} = r\, e^{i(\theta + 2\pi n)}$

$r = \sqrt{x^2 + y^2}$ modulus

$\theta = \tan^{-1} \dfrac{y}{x}$ argument

$\rightarrow$ $e^{i\theta} = \cos\theta + i \sin\theta$

Multi-valued function $\rightarrow$ single-valued in each branch

E.g., $z^{1/m} = r^{1/m}\, e^{i(\theta + 2\pi n)/m}$ has *m* branches.

$\ln z = \ln r + i(\theta + 2\pi n)$ has an infinite number of branches.

Limit of a function, continuity and differentiability

- The limit of f(z) as z approaches z_0 is w_0.

$$\lim_{z \to z_0} f(z) = w_0$$

- f(z) is said to be continuous at $z=z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- Let f(z) be a single valued function of the variable z,then

$$f'(z) = \lim_{z \to z_0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

  Provided limit exists and is independent of the path along which $\delta z \to 0$

Analytic Function

- A function f(z) is said to be analytic at a point z_0 ,if f(z) is differentiable not only at z_0 but at every point of some neighbourhood of z_0.
- A point where the function ceases to be analytic is called a singular point.
- Analytic function is always differentiable and continuous.But converse not true.

Necessary condition for f(z) to be analytic

- The necessary conditions for a function f(z)=u+iv to be analytic at all the points in a region R are:

i. $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

ii. $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$ provided u_x, u_y, v_x, v_y exist.

conditions (i) and (ii) also called Cauchy Riemann equations.

2. Cauchy Reimann Conditions

$z = x + i\,y$

Derivative : $$\frac{d\,f(z)}{d\,z} = f'(z) = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

where limit is independent of path of $\delta z \to 0$.

Let $$f(z) = u(z) + i\,v(z)$$

$\rightarrow$ $$\delta z = \delta x + i\,\delta y \qquad \delta f = \delta u + i\,\delta v \qquad \rightarrow \qquad \frac{\delta f}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

$\delta z = \delta x \quad \rightarrow$ $$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\,\frac{\delta v}{\delta x} \right) = \frac{\partial\,u}{\partial\,x} + i\,\frac{\partial\,v}{\partial\,x}$$

$\delta z = \delta y \quad \rightarrow$ $$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i\,\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i\,\frac{\partial\,u}{\partial\,y} + \frac{\partial\,v}{\partial\,y}$$

$\therefore$ f' exists $\rightarrow$ $$\frac{\partial\,u}{\partial\,x} = \frac{\partial\,v}{\partial\,y} \quad \& \quad \frac{\partial\,u}{\partial\,y} = -\frac{\partial\,v}{\partial\,x}$$ Cauchy- Reiman equations

$$z = x + i\,y \qquad f(z) = u(z) + i\,v(z)$$

f' exists $\rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ Cauchy- Reimann Conditions

$$f(z) = f(x,y) \quad \rightarrow \quad \delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right)\delta y$$

If the CRCs are satisfied, $\delta f = \left(\dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x}\right)(\delta x + i\,\delta y)$

$\rightarrow$ $\dfrac{\delta f}{\delta z} = \dfrac{\partial u}{\partial x} + i\,\dfrac{\partial v}{\partial x}$ is independent of path of $\delta z \rightarrow 0.$

i.e., f' exists $\leftrightarrow$ CRCs satisfied.

C-R EQUATIONS IN POLAR FORM

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Proof do yourself

Example 11.2.1. z^2 is Analytic

$z = x + i y$

$$f(z) = z^2 = x^2 - y^2 + 2i\,xy = u + i\,v$$

$$\rightarrow \quad \begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned} \quad \rightarrow \quad \frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$\therefore\ f'$ exists & single-valued $\forall$ finite z.

i.e., z^2 is an entire function.

Example 11.2.2. z^* is Not Analytic

$z = x + iy$

$$f(z) = z^* = x - iy \quad = u + iv$$

$$\rightarrow \qquad \begin{array}{l} u = x \\ v = -y \end{array} \qquad \rightarrow \qquad \frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

$\therefore \; f'$ doesn't exist $\forall \, z$, even though it is continuous every where.

i.e., z^2 is nowhere analytic.

Harmonic Functions

CRCs $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

By definition, derivatives of a real function f depend only on the local behavior of f.

But derivatives of a complex function f depend on the global behavior of f.

Let $\psi(z) = u + iv$

ψ is analytic $\rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \qquad \rightarrow \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

i.e., The real & imaginary parts of ψ must each satisfy a 2-D Laplace equation.

(u & v are harmonic functions)

CRCs $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Contours of u & v are given by $\quad u(x,y) = c \qquad v(x,y) = c'$

$$\rightarrow \qquad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0 \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0$$

Thus, the slopes at each point of these contours are

$$m_u = \left(\frac{dy}{dx} \right)_u = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} \qquad m_v = \left(\frac{dy}{dx} \right)_v = -\frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}$$

CRCs $\rightarrow \quad m_u m_v = -1 \quad$ at the intersections of these 2 sets of contours

i.e., these 2 sets of contours are orthogonal to each other.

(u & v are complementary)

Method to find the conjugate function

- If $f(z)=u+iv$ and u is known.
- To find v,conjugate function.
- Method:

We know $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

using Cauchy Riemann equations, replace v_x by $-u_y$ and v_y by u_x

$$dv = \frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$v = -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

Method to find conjugate function

- V=v(x,y) is given we need to find u.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

using Cauchy Riemann equations, replace

u_y by $-v_x$ by and u_x by v_y

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$u = \int \frac{\partial v}{\partial y} dx - \int \frac{\partial v}{\partial x} dy$$

Problems

Q.Let f(z)=u(x,y)+iv(x,y) be an analytic function.If u=3x-2xy,find v and express f(z) in terms of z.

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = \frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$v = - \int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$$

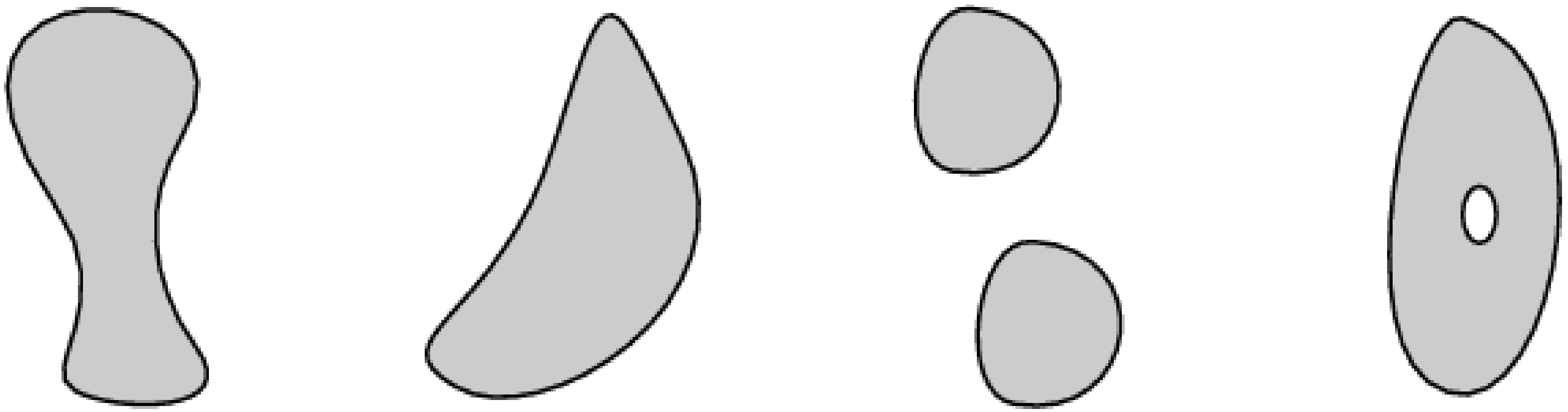
dv=2xdx+(3-2y)dy

$v=\int 2xdx+\int(3-2y)dy=x^2+3y-y^2+c$

$f(z)=iz^2+3z+ic$

Defn: A ***simply connected domain*** D is a domain such that every simple closed contour within it encloses only points of D.

The set of points interior to a simply closed contour is an example.



simply connected simply connected not simply connected not simply connected

A domain that is not simply connected is said to be *multiply connected* for example, the annular domain between two concentric circles.

The Cauchy – Goursat theorem for a simply connected domain D is as follows:

Theorem: If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

for every closed contour C lying in D.

Result: Let C_1 and C_2 denote positively oriented simple closed contours, where C_2 is interior to C_1.

If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Ex.1 Evaluate

$$\int_C f(z)\,dz$$

when $f(z) = ze^{-z}$,

C: $|z|=1$.

Ans: 0 (Why??)

Ex.2 ***Evaluate***

$$\int_C f(z)\,dz$$

when

$$f(z) = \frac{z^2 \sin z}{z-4}, \quad C: |z| = 2.$$

Ans: 0 (Why??)

<u>Qs 3/154.</u> Let C_0 denote the circle $|z - z_0| = R$, taken counter clockwise using the parametric representation

$$z = z_0 + \mathrm{R}e^{i\theta} \ \left(-\pi \leq \theta \leq \pi\right)$$

for C_0 to derive the following integrations:

$$(a)\quad \int_{C_0} \frac{dz}{z - z_0} = 2\pi i$$

$$(b)\quad \int_{C_0} (z - z_0)^{n-1} dz = 0,\ n = \pm 1, \pm 2, \ldots$$

$$(c)\quad \int_{C_0} \left(z - z_0\right)^{a-1} dz = \frac{2iR^{a}}{a} \sin\left(a\pi\right),$$

where $a \neq 0$ is any real no.

Sol. We have $\left|z - z_0\right| = R$

$$\Rightarrow z - z_0 = \mathrm{R}e^{i\theta}$$

$$\Rightarrow dz = \mathrm{R}e^{i\theta}.id\theta$$

a)

$$I = \int_{C_0} \frac{dz}{z - z_0} = \int_{-\pi}^{\pi} \frac{\mathrm{R}e^{i\theta}.id\theta}{\mathrm{R}e^{i\theta}}$$

$$= i\left(\pi - \left(-\pi\right)\right) = 2\pi i$$

b)

$$I = \int_{C_0} \left(z - z_0\right)^{n-1} dz$$

$$= \int_{-\pi}^{\pi} R^{n-1} e^{i(n-1)\theta} . \mathrm{Re}^{i\theta} d\theta$$

$$= 0 \quad \text{(after simplification)}$$

c)

$$I = \int_{C_0} (z - z_0)^{a-1} dz$$

$$= \int_{-\pi}^{\pi} R^{a-1} e^{i(a-1)\theta} . \mathrm{Re}^{i\theta} d\theta$$

$$= \frac{2i\, R^a}{a} Sin(a\pi)$$

Exercise:

- Does Cauchy – Goursat Theorem hold separately for the real or imaginary part of an analytic function f(z) ? Justify your answer.

Cauchy Integral Formula

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sence, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}.$$

Derivative Formula

Suppose that a function f is analytic everywhere inside and on a simple closed contour C, taken in the positive sence. If z_0 is any point interior to C, then

$$a) \quad f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^2},$$

$$b) \quad f''(z_0) = \frac{(2)!}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^3},$$

$c)$

$$f^{(n)}(z_0) = \frac{(n)!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}.$$

Theorem:

If $f(z)$ is analytic at z_0, then its derivatives of all orders exist at z_0 and are themselves analytic at z_0.

Qs.1(a)/163: Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the following integral $\displaystyle\int_C \frac{\cos z \, dz}{z(z^2+8)}$.

Ans : $\boldsymbol{\pi i/4}$.

Qs. 2(b)/163: Find the value of the integral of g(z) around the circle $|z-i|=2$ in the positive sense when

$$g(z)=\frac{1}{\left(z^{2}+4\right)^{2}}.$$

$$Sol:\int_C \frac{dz}{\left(z^2+4\right)^2}=\int_C \frac{dz}{\left(z+2i\right)^2\left(z-2i\right)^2}$$

$$=2\pi i\frac{d}{dz}\left(\frac{1}{\left(z+2i\right)^2}\right)_{z=2i}$$

$$=\frac{\pi}{16}$$

<u>Qs.4/163:</u> Let C be any simple closed contour, described the positive sense in the z- plane and write

$$g(w) = \int_C \frac{z^3 + 2z}{(z-w)^3} dz$$

Show that

$$g(w) = 6\pi i w$$

when w is inside C and that

$$g(w) = 0$$

when w is outside C.

Case I : Let w is inside C.

Let $f(z) = z^3 + 2z.$ Then

$$g(w) = \int_C \frac{f(z)}{(z-w)^3} dz,$$

$$= \frac{2\pi i}{2} f''(w)$$

$$f(z) = z^3 + 2z$$

$$\Rightarrow f'(z) = 3z^2 + z$$

$$\Rightarrow f''(z) = 6z$$

$$\Rightarrow f''(w) = 6w$$

$$\therefore I = g(w) = 6\pi i w$$

Case 2. When w is outside C, then by Cauchy Goursat Theorem $g(w) = 0$.

Qs. 5/163: Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)}{(z-z_0)}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

Sol. Let

$$I_1 = \int_C \frac{f'(z)}{(z - z_0)} dz \text{ and}$$

$$I_2 = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

Case I: Let z_0 is inside C, then

$$I_1 = \int_C \frac{f'(z)}{(z - z_0)} dz = 2\pi i\, f'(z)\Big|_{z=z_0}$$

$$= 2\pi i\, f'(z_0)$$

and

$$I_2 = \int_C \frac{f(z)}{(z - z_0)^2} dz$$

$$= 2\pi i\, f'(z_0)$$

$$\therefore I_1 = I_2 .$$

Case II: Let z_0 is outside C

Then $I_1 = I_2 = 0.$

(WHY ???)

Morera’s Theorem:

If a function $f(z)$ is continuous throughout in a domain D and if

$$\int_C f(z)dz = 0,$$

for every closed contour C lying in D, then $f(z)$ is analytic in D.

LIOUVILLE’S THEOREM

If f is entire and bounded in the complex plane, then f(z) is constant throughout the plane.

Theorem: Suppose that

(i) C is a simple closed contour, described in the counter-clockwise direction,

(ii) C_k (k = 1, 2, …., n) are finite no. of simple closed contours, all described in the clockwise direction, which are interior to C and whose interiors are disjoin.

If $f(z)$ is analytic throughout the closed region consisting of all points within and on C except for the points interior to C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Ex. Evaluate $\int_C \frac{dz}{z(z^2+1)}$ for all possible choices of the contour C that does not pass through any of the points 0, $\pm i$.

Solution:

Case 1. Let C does not enclose 0, $\pm i$.

Then

$$I = \int_C \frac{dz}{z^2+1} = 0 \;\; \textit{by CG Theorem.}$$

Case 2a. Let C encloses only 0.

Then $I=\int_C \frac{dz}{z(z^2+1)}$

$$=\int_C \frac{f(z)dz}{z-0},\ f(z)=\frac{1}{(z^2+1)}$$

$$=2\pi i\, f(0)$$

$$=2\pi i$$

Exercise:

Case 2b. Let C encloses only i.

Ans: $I = -\pi i$

Case 2c. Let C encloses only -i.

Ans: $I = -\pi i$

Case 3 a). Let C encloses only 0, -i.

then

$$I = \int_{C_0} \frac{dz}{z(z+i)(z-i)} + \int_{C_{-i}} \frac{dz}{z(z+i)(z-i)}$$

where C_0 and C_{-i} are sufficiently small circles around 0 and –i resp.

$$= \int_{C_0} \frac{\dfrac{1}{(z+i)(z-i)}}{z}\,dz$$

$$+ \int_{C_{-i}} \frac{\dfrac{1}{z(z-i)}}{(z+i)}\,dz$$

$$= (2\pi i)\left(-\frac{1}{i^2}\right) + (2\pi i)\left(\frac{1}{-i(-2i)}\right)$$

$$= \pi i$$

Case 3 b). Let C encloses only 0, i.

then

$$I = \int_{C_0} \frac{dz}{z(z+i)(z-i)} + \int_{C_i} \frac{dz}{z(z+i)(z-i)}$$

where C_0 and C_i are sufficiently small circles around 0 and i resp.

$$I = \int_{C_0} \frac{\dfrac{1}{(z+i)(z-i)}}{z}\,dz + \int_{C_i} \frac{\dfrac{1}{z(z+i)}}{(z-i)}\,dz$$

$$= 2\pi i + (2\pi i)\left(\frac{1}{i.2i}\right)$$

$$= \pi i$$

Case 3 c). Let C encloses only -i, +i. Then

$$I = \int_{C_i} \frac{\frac{1}{z(z+i)}}{z-i} dz + \int_{C_{-i}} \frac{\frac{1}{z(z-i)}}{(z+i)} dz$$

$$= (2\pi i)\left(\frac{1}{i.2i}\right) + (2\pi i)\left(\frac{1}{-i.-2i}\right)$$

$$= -2\pi i$$

Case 3 d). Let C encloses all of the points 0, -i, +i.

Then

$$I = \int_{C_0} \frac{\frac{1}{z^2+1}}{z} dz + \int_{C_i} \frac{\frac{1}{z(z+i)}}{z-i} dz$$

$$+ \int_{C_{-i}} \frac{\frac{1}{z(z-i)}}{(z+i)} dz$$

$$= 2\pi i - \pi i - \pi i$$

$$= 0$$

$$I = \int_{C_0} \frac{\frac{1}{z^2+1}}{z} dz + \int_{C_i} \frac{\frac{1}{z(z+i)}}{z-i} dz$$

$$+ \int_{C_{-i}} \frac{\frac{1}{z(z-i)}}{(z+i)} dz$$

$$= 2\pi i - \pi i - \pi i$$

$$= 0$$

Taylor's Theorem: Suppose that a function f(z) is analytic throughout a disk $|z - z_0| < R_0$ centered at z_0 and with radius R_0. Then f(z) has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad (|z - z_0| < R_0)$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2 \ldots .)$$

Maclaurin Series

Taylor Series about the point $z_0 = 0$ is called Maclaurin series, i.e.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad (|z| < R_0)$$

Examples:

1. $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (|z| < \infty)$$

2. $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

$$(|z| < \infty)$$

3. $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

$$(|z| < \infty)$$

4. $$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!},$$

$$(|z| < \infty)$$

5. $$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \qquad (|z| < \infty)$$

6. $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad (|z| < 1)$$

7. $$\frac{1}{1+z}=\sum_{n=0}^{\infty}(-1)^n z^n, \qquad (|z|<1)$$

Laurent's Theorem: Suppose that a function f(z) is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centered at z_0 and let C denote any positively oriented simple closed contour around z_0 and lying in that domain.

Then, at each point in domain f(z) has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$\left(R_1 < |z - z_0| < R_2\right)$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = 0,1,2.....)$$

and

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \quad (n = 0,1,2,...)$$

Example:

Find the Laurent series representation of

$$f(z) = \frac{z}{(z-1)(z-3)}$$

when

$$(a)\ D_1 : 0 < |z| < 1,$$

$$(b)\ D_2 : 1 < |z| < 3,$$

$$(c)\ D_3 : 3 < |z| < \infty,$$

We have

$$f(z) = \frac{z}{(z-1)(z-3)}$$

$$= -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

(a) Consider the domain

$$D_1 : 0 < |z| < 1.$$

Then $f(z)$ is analytic in D_1.

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$= \frac{1}{2(1-z)} - \frac{3}{2\times 3(1-\frac{z}{3})}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$\Rightarrow f(z) = \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(1 - \frac{1}{3^n}\right) z^n$$

(b) Consider the domain

$$D_2 : 1 < |z| < 3.$$

Then $f(z)$ is analytic in D_2.

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$= -\frac{1}{2z(1-\frac{1}{z})} - \frac{3}{2\times 3(1-\frac{z}{3})}$$

$$= -\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$\Rightarrow f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n}$$

(c) Consider the domain

$$D_3 : 3 < |z| < \infty.$$

Then $f(z)$ is analytic in D_3.

Note that

$$\frac{1}{|z|} < \frac{3}{|z|} < 1.$$

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$= -\frac{1}{2z(1-\frac{1}{z})} + \frac{3}{2\times z(1-\frac{3}{z})}$$

$$= -\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{3}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

$$\Rightarrow f(z) = -\frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n} + \frac{3}{2z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^{n}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^{n+1}}{z^{n+1}}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(1-3^{n+1}\right)\frac{1}{z^{n+1}}.$$

Excercise:

Show that, when $0 < |z-1| < 2$,

the Laurent series representation

of

$$f(z) = \frac{z}{(z-1)(z-3)}$$

is

$$f(z) = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}} - \frac{1}{2(z-1)}.$$

RESIDUE

(1) Consider a function f(z) &

$$\text{let } z = \frac{1}{w}. \quad \text{Then}$$

$$f(z) = f\left(\frac{1}{w}\right) = g(w)$$

(i) $f(z)$ is said to be analytic at infinity if $g(w)$ is analytic at $w = 0$.

(ii) $f(z)$ is said to be singular at infinity if $g(w)$ is singular at $w = 0$.

(2) Zero of an analytic function:

Let $f(z)$ is analytic in a domain D.

If $f(z_0) = 0$ for some $z = z_0$, *then*
$z = z_0$ is called zero of $f(z)$.

If $f(z_0) = f'(z_0) = f''(z_0) = \ldots.$

$$= f^{(n-1)}(z_0) = 0, \text{ but}$$

$f^{(n)}(z_0) \neq 0$, then $z = z_0$ is called ZERO OF ORDER n of $f(z)$.

$i.e.\quad z = z_0$ is called zero of order n of f(z) if

$$f(z) = (z - z_0)^n g(z),$$

$$where \quad g(z_0) \neq 0.$$

(3) Singular Point of a fn f(z):

(i) If a function f(z) fails to be analytic at a point z_0, but it is analytic at some point in every nbd of z_0, then z_0 is called Singular Point of f(z).

(ii) Isolated Singularity

The point z_0 is called an isolated singularity of $f(z)$ if

(a) z_0 is a singular point of $f(z)$

(b) $f(z)$ is analytic in a deleted nbd

$$N : 0 < |z - z_0| < \in .$$

(4) (i) Let z_0 is an isolated singularity of $f(z)$

$\Rightarrow \exists\, R > 0$ such that $f(z)$ is analytic in $0 < |z - z_0| < R$.

Hence f(z) has Laurent series expansion :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n},$$

$$0 < |z - z_0| < R$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_c \dfrac{f(z)dz}{(z-z_0)^{n+1}}$,

$$b_n = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{(z-z_0)^{-n+1}},$$

C is any positively oriented simple closed contour around z_0 *and* lying in the puctured disc $0 < |z - z_0| < R$.

$(ii)\ \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called principal part (PP) of the Laurent series, i.e.

$$PP = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

$$= \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots\ldots\ldots$$

If $b_k \neq 0$, for some k, say $k = m$,

and $b_n = 0 \quad \forall n > m$, then

$$PP = \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)} + \ldots + \frac{b_m}{(z - z_0)^m}$$

Then the singularity $z = z_0$ of $f(z)$ is called POLE OF ORDER m.

If $m = 1$, then z_0 is a pole of order 1 and is called a SIMPLE POLE.

(iii) If an analytic function $f(z)$ has a singularity other than a pole, then this singularity is known as ESSENTIAL SINGULARITY of $f(z)$, i.e.

if $b_n \neq 0$ for infinitely many n,

then the singularity z_0 is called ESSENTIAL SINGULARITY of f(z).

(iv) If $b_n = 0 \quad \forall n$,

then the singularity z_0 is called REMOVABLE SINGULARITY of f(z).

RESIDUE :

The PP of the Laurent series is given by

$$PP = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \qquad \text{where}$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}$$

If $n = 1$, then

$$b_1 = \frac{1}{2\pi i} \int_c f(z)\, dz$$

is called RESIDUE of f(z)

at $z = z_0$, and we write

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

$$= \text{coeff of } \frac{1}{z - z_0}$$

Residue Theorem:

Let C be a positively oriented simple closed contour. Suppose that f(z) is analytic within and on C except for a finite number of singular points $z_k \ (k = 1, 2, \ldots. n)$ inside C.

Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \left(\operatorname*{Res}_{z=z_k} f(z) \right)$$

How to find residue of a given fn f(z):

$Ex1:$ Let $f(z)=\dfrac{\sin z}{z^4},\quad 0<|z|<\infty.$

Now $f(z)=\dfrac{1}{z^4}(\sin z)$

$$=\frac{1}{z^4}\left(z-\frac{z^3}{(3)!}+\frac{z^5}{(5)!}-\frac{z^7}{(7)!}+....\right)$$

$$f(z) = \frac{1}{z^3} - \frac{1}{(3)!}.\frac{1}{z} + \frac{1}{(5)!}.z - \frac{1}{(7)!}z^3 + ...$$

$$0 < |z| < \infty$$

$$PP = -\frac{1}{(3)!}.\frac{1}{z} + \frac{1}{z^3}$$

Note that $z = 0$ is a pole of order ???

Hence

$$\operatorname*{Res}_{z=0} f(z) = b_1 = coeff\ of\ \frac{1}{z} = -\frac{1}{6}$$

$$\therefore \int_{c:|z|=1} \frac{\sin z}{z^4} dz = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

$$= -\frac{\pi i}{3}$$

Ex 2. Find the residue of $f(z) = \exp(1/z)$, and hence evaluate

$$\int_C f(z)dz, \quad C : |z| = 1.$$

Soln :

$$f(z) = \exp\left(\frac{1}{z}\right)$$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \ldots..$$

Note: $z = 0$ is an essential singularity of $f(z)$.

$$\Rightarrow b_1 = \text{coeff of } \frac{1}{z}$$

$$= \operatorname*{Res}_{z=0} f(z)$$

$$= 1$$

Hence

$$\int_C f(z)dz = 2\pi i.$$

Ex 3. Find the residue of

$$f(z) = \exp(1/z^2),$$ and

hence evaluate

$$\int_C f(z)dz, \quad C: |z| = 1.$$

Hints:

1. $z = 0$ is an essential singularity of f(z).

2. $b_1 = \operatorname*{Res}_{z=0} f(z) = 0.$

3. $I = 0.$

How to find the residues ?

We have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

Case IA : Let $z = z_0$ is a simple pole of $f(z)$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0}$$

$$\Rightarrow (z - z_0)\ f(z)$$

$$= b_1 + (z - z_0)\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$\Rightarrow \lim_{z\to z_0} (z - z_0)\ f(z) = b_1$$

$$= \operatorname*{Res}_{z=z_0}\ f(z)$$

Case IB : Let f(z) has a simple pole at $z = z_0$ and f(z) is of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where

(i) $p(z)$ & $q(z)$ are analytic

at $z = z_0$,

(ii) $p(z_0) \neq 0$, and

(iii) $q(z)$ has a simple zero

at $z = z_0$,

Then

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)}$$

$$= \frac{p(z_0)}{q'(z_0)}$$

Case II : Let z_0 be a pole of order $m > 1$ for the function f(z).

$$\text{Then } f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$+ \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_m}{(z - z_0)^m}$$

$$\Rightarrow (z - z_0)^m \ f(z)$$

$$= (z - z_0)^m \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$+ b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2}$$

$$+ \ldots\ldots + b_{m-1} (z - z_0) + b_m$$

$$Let\ \phi(z) = (z - z_0)^m f(z)$$

then

$$\underset{z=z_0}{\text{Res}} f(z) = b_1$$

$$= \text{coeff. of } (z - z_0)^{m-1} \text{ in the}$$

$$\text{expansion of } \phi(z)$$

$$= \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \text{ by Taylor's Thm}$$

Thus if z_0 *is* a pole of order $m > 1$ of $f(z)$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\phi^{m-1}(z)\right]$$

$$\operatorname*{Res}_{z=z_0} f(z)$$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left(z - z_0\right)^m f(z) \right]$$

Ex1.

Find the residue of $f(z)$ at $z=0$ and $z=-1$, where

$$f(z)=\frac{1}{z+z^2}.$$

Soln :

Note that

$$z = 0 \quad \text{and} \quad z = -1$$

are simple poles of f(z).

$$\therefore \operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0} (z-0) f(z)$$

$$= \lim_{z \to 0} \left(\frac{1}{1+z} \right) = 1$$

$$\therefore \operatorname*{Res}_{z=-1} f(z) = \lim_{z\to 0} (z+1) f(z)$$

$$= \lim_{z\to -1} \left(\frac{1}{z}\right) = -1.$$

$Q.2$ (a) Evaluate $I = \int_{c:|z|=3} \frac{e^{-z}}{z^2} dz.$

Soln :

Clearly, $z = 0$ is a pole of order 2

of $f(z) = \frac{e^{-z}}{z^2}.$

Now

$$I = \int_{c:|z|=3} f(z)\,dz$$

$$= 2\pi i \sum_{z=z_k} \operatorname{Res} f(z),$$

$$f(z) = \frac{e^{-z}}{z^2}$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \cdot \lim_{z \to 0} \left[\frac{d}{dz} \left(z^2 f(z) \right) \right]$$

$$= \lim_{z \to 0} \left[\frac{d}{dz} e^{-z} \right]$$

$$\Rightarrow \operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0} \left(-e^{-z} \right)$$

$$= -1$$

$$\therefore I = -2\pi i$$

Q.2 (b) Evaluate

$$\mathrm{I} = \int_{c:|z-3|=1} \frac{e^{-z}}{z^2}\, dz.$$

Ans: $\mathrm{I} = 0$ (WHY ???)

Ex2(c). Evaluate

$$I = \int_{c:|z|=3} \frac{e^{-z}}{(z-1)^2}.$$

*So*ln :

$z = 1$ is pole of order 2 of

$$f(z) = \frac{e^{-z}}{(z-1)^2}.$$

$$\therefore \operatorname*{Res}_{z=1} f(z) = \frac{d}{dz}\left(e^{-z}\right)\Big|_{z=1}$$

$$= -e^{-z}\Big|_{z=1} = -\frac{1}{e}$$

$$\therefore I = -\frac{2\pi i}{e}$$

$$(c)\ I = \oint_{|z|=3} z^2 . e^{\frac{1}{z}}\ dz$$

$$\text{Let}\quad f(z) = z^2 e^{\frac{1}{z}}$$

$$\Rightarrow z = 0\ is\ \text{an essential singularity of}\ f(z)$$

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2!}.\frac{1}{z^2} + \frac{1}{3!}.\frac{1}{z^3} + \frac{1}{4!}\frac{1}{z^4} + \ldots\right)$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!}.\frac{1}{z} + \frac{1}{4!}.\frac{1}{z^2} + \ldots.$$

$$\therefore \operatorname*{Res}_{z=0} f(z) = \text{coeff. of } \frac{1}{z} = \frac{1}{6}$$

$$\therefore I = 2\pi i \times \frac{1}{6} = \frac{\pi i}{3}$$

(d)

$$I = \int_{|z|=3} \frac{z+1}{z^2 - 2z}\, dz$$

$$\text{Let } f(z) = \frac{z+1}{z^2 - 2z} = \frac{z+1}{z(z-2)}$$

$$\Rightarrow z = 0 \quad \& \; z = 2 \text{ are simple poles}$$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} z\, f(z)$$

$$= \lim_{z\to 0} \frac{z+1}{z-2}$$

$$= -\frac{1}{2}$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z \to 2} (z-2) f(z)$$

$$= \frac{3}{2}$$

$$\therefore I = 2\pi i \sum \operatorname{Res} f(z)$$

$$= 2\pi i \left(-\frac{1}{2} + \frac{3}{2} \right) = 2\pi i.$$

$Q.3, p.233$

Let $f(z)$ be analytic at z_0, and consider

$$g(z) = \frac{f(z)}{z - z_0}.$$

Then Show that

(a) If $f(z_0) \neq 0$,

then z_0 is a simple pole

of $g(z)$ and

$$\operatorname{Res}_{z=z_0} g(z) = f(z_0)$$

(b) If $f(z_0) = 0$,

then z_0 is a

removable singularity of $g(z)$

and $\operatorname*{Res}_{z=z_0} g(z) = 0$.

$Sol : \because f(z)$ is analytic at z_0

$\Rightarrow f(z)$ has Taylor's series expansion about z_0, &

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2!} + (z - z_0)^3 \frac{f'''(z_0)}{3!} + \ldots$$

$$
\Rightarrow g(z) = \frac{f(z)}{z - z_0}
$$

$$
= \frac{f(z_0)}{z - z_0} + f'(z_0)
$$

$$
+ (z - z_0)\frac{f''(z_0)}{2!}
$$

$$
+ (z - z_0)^2 \frac{f'''(z_0)}{3!} + \dots
$$

(a) Clearly if $f(z_0) \neq 0$, Then principal part (P.P) of g(z) is

$$= \frac{f(z_0)}{z - z_0}$$

$\therefore z_0$ is a simple pole of $g(z)$

and

$$\operatorname*{Res}_{z=z_0} g(z) = b_1 = \text{coeff of } \frac{1}{z - z_0}$$

$$= f(z_0)$$

(b) If $f(z_0) = 0$, then p.p. of $g(z)$ is 0

$\Rightarrow b_n = 0 \ \forall n$

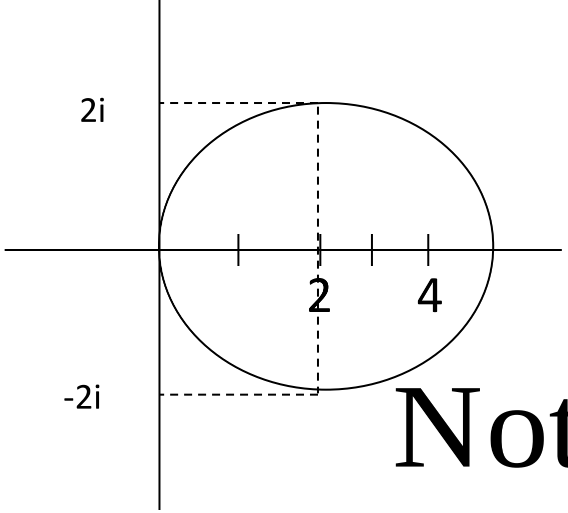
$\Rightarrow z = z_0$ is a removable singularity of $g(z)$, and

$$\operatorname*{Res}_{z=z_0} g(z) = 0$$

$$Q.4\,(a)\ \mathrm{I} = \int_{c} \frac{3z^3 + 2}{(z-1)(z^2+9)},\ c: |z-2| = 2$$

$$\text{Let} \quad f(z) = \frac{3z^3 + 2}{(z-1)(z^2+9)}$$

Then $1, 3i, -3i$ are simple poles of f(z)



Note: z = 1 is only inside C

$$\therefore \operatorname*{Res}_{z=1} f(z) = \left[\frac{3z^3+2}{z^2+9}\right]\Bigg|_{z=1}$$

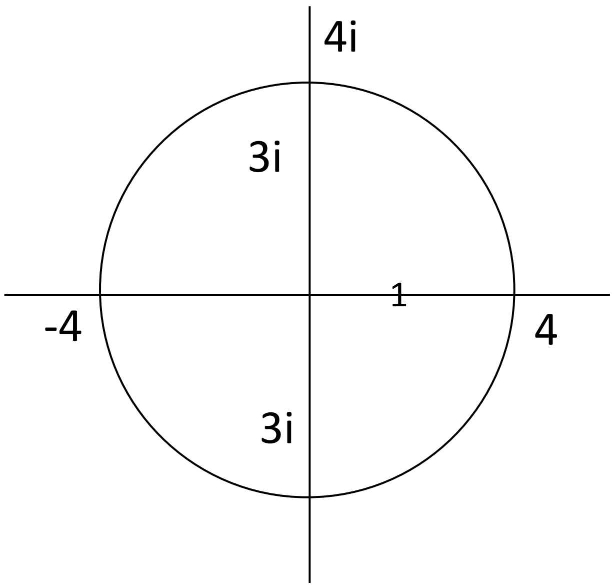
$$= \frac{5}{10} = \frac{1}{2}$$

$$\therefore I = 2\pi i \times \operatorname*{Res}_{z=1} f(z) = \pi i$$

$$(b)\ \mathrm{c} : |z| = 4$$

$$\textit{Then } 1, 3i, -3i \text{ are all inside C}$$

$$\therefore \operatorname*{Res}_{z=1} f(z) = \frac{1}{2}$$



$$\operatorname*{Res}_{z=3i} f(z) = \frac{3z^3+2}{(z-1)(z+3i)}\bigg|_{z=3i}$$

$$= \frac{-81i+2}{(3i-1)(6i)}$$

$$= \frac{2-81i}{-18-6i}$$

$$\operatorname*{Res}_{z=-3i} f(z) = \frac{3z^3+2}{(z-1)(z-3i)}\bigg|_{z=-3i}$$

$$= \frac{+81i+2}{(-3i-1)(-6i)}$$

$$= \frac{2+81i}{-18+6i}$$

$$\therefore \sum \operatorname{Res} f(z)$$

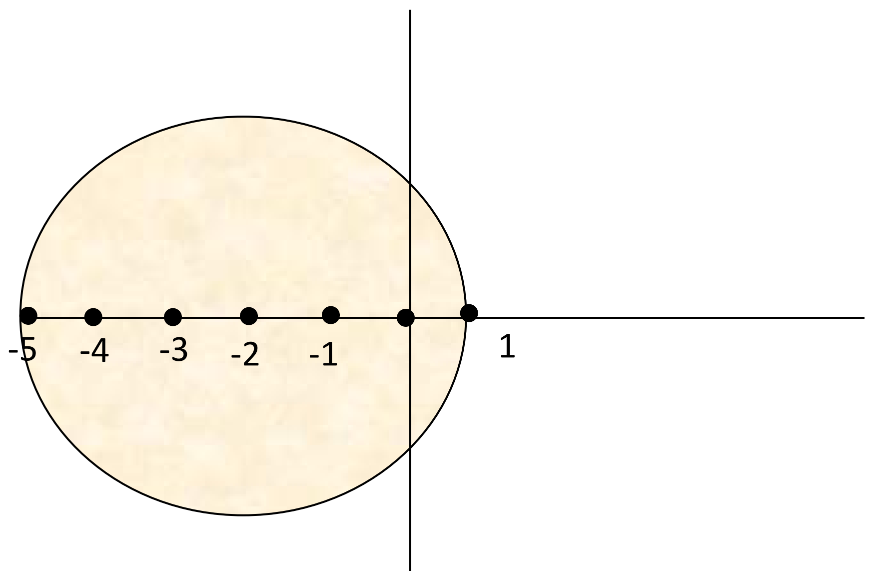
$$= \frac{1}{2} + \frac{2+81i}{6i-18} - \frac{2-81i}{6i+18}$$

$$= 3$$

$$\therefore I = 2\pi i \sum \operatorname{Res} f(z) = 6\pi i$$

$$Q.5(b)\ I = \int_c \frac{dz}{z^3(z+4)}, c: |z+2| = 3$$

Let $$f(z) = \frac{1}{z^3(z+4)}$$



$\Rightarrow z = 0$ is a pole of order 3 and

$z = -4$ is a simple pole

& both lie inside C.

$$\therefore \operatorname*{Res}_{z=0} f(z) = \frac{1}{2} \cdot \frac{d^2}{dz^2}\left[\frac{1}{z+4}\right]\Bigg|_{z=0}$$

$$= \frac{1}{4^3}$$

$$\operatorname*{Res}_{z=-4} f(z) = \frac{1}{z^3}\bigg|_{z=-4} = -\frac{1}{4^3}$$

$$\therefore I = 2\pi i\left(\frac{1}{4^3} - \frac{1}{4^3}\right) = 0$$