DEPARTMENT OF MATHEMATICS

BIRLA INSTITUTE OF TECHNOLOGY MESRA, RANCHI

MA107 Mathematics-II, Session: (SP-2020)

Tutorial - 4 (Module IV)

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1. Show that each of these functions is nowhere analytic

   a) $f(z) = \bar{z}$ b) $f(z) = |z|^2$ c) $f(z) = xy + iy$

   d) $f(z) = 2xy + i(x^2 - y^2)$ e) $f(z) = e^{y+ix}$

2. Verify whether the function $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is non-analytic at $z = 0$.

3. Let $u(x, y)$ and $v(x, y)$ denote the real and imaginary components of the function $f(z)$ defined by

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

   Verify that the Cauchy-Riemann equations are satisfied at the origin $z = (0, 0)$.

4. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

   a) $u(x, y) = x^2 - y^2 - y$ b) $u(x, y) = x^3 - 3x^2y - 3xy^2 - y^3$

5. Apply Cauchy's theorem to show that $\int f(z)dz = 0$ when the contour C is the unit circle $|z| = 1$, in either direction, and when

   a) $f(z) = \dfrac{z^2}{z-3}$ b) $f(z) = ze^{-z}$ c) $f(z) = \dfrac{1}{z^2+z+2}$

6. Find the value of the integral

   a) $\displaystyle\int_{|z|=1} \frac{zsinz}{z-\pi}dz$ b) $\displaystyle\int_{|z|=1} \frac{1}{z+2}dz$ c) $\displaystyle\int_{|z|=1} \frac{e^z}{z^3}dz$

   d) $\displaystyle\int_{|z|=2} \frac{1}{z^2(z-3)}dz$ e) $\displaystyle\int_{|z|=2} \frac{z^3+2z+1}{(z+1)^3}dz$

7. Find the radius of convergence of the following power series

   a) $\displaystyle\sum_{n=0}^{\infty}(1+\frac{1}{n})^{n^2}z^n$ b) $\displaystyle\sum_{n=0}^{\infty} n!z^n$ c) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}z^n$

8. Show that when $0 < |z-1| < 2$, $\dfrac{z}{(z-1)(z-3)} = -3\displaystyle\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$.

9. Find the residue at $z = 0$ of the function

   a) $\dfrac{1}{z+z^2}$ b) $\dfrac{z - \sin z}{z^3}$ c) $\dfrac{e^z}{z^3}$ d) $e^{\frac{1}{z}}$

10. Use Cauchy's residue theorem to evaluate the integrals of each of these functions around the circle $|z| = 3$ in the positive sense

a) $\dfrac{e^{-z}}{z^2}$　　b) $\dfrac{z+1}{z(z-2)}$　　c) $z^2 e^{\frac{1}{z}}$

Solution:

4. a) $v(x, y) = 2xy + x$　　b) $v(x, y) = x^3 + 3x^2y + 3xy^2 - y^3$

6. a) 0　　b) 0　　c) πi　　d) $\dfrac{-2\pi i}{9}$　　e) $-6\pi i$

7. a) e　　b) 0　　c) ∞　　d) e

9. a) 1　　b) $\dfrac{1}{6}$　　c) $\dfrac{1}{2}$　　d) 1

10. a) $-\pi i$　　b) $-2\pi i$　　c) $\dfrac{\pi i}{3}$