DEPARTMENT OF MATHEMATICS

BIRLA INSTITUTE OF TECHNOLOGY MESRA, RANCHI

MA107 Mathematics-II, Session: (SP-2020)

Tutorial - 3 (Module III)

---

1. Find the Fourier series of $f(x) = \pi^2 - x^2$ in $-\pi < x < \pi$.

Solution:

$$a_n = -\frac{4}{n^2}\cos(n\pi), \ n = 1, 2, 3, \ldots$$

$$a_0 = \frac{4}{3}\pi^2 \text{ and } b_n = 0, \ n = 1, 2, 3, \ldots$$

2. Find the Fourier series for the function $f(x)$ if $f(x)$ is defined in $-\pi < x < \pi$ as

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

Hence deduce that

$$\frac{1}{4}(\pi - 2) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

Solution:

$$a_n = \begin{cases} \frac{2}{\pi}\frac{1}{(1+n)(1-n)} & \text{n is even} \\ 0 & \text{n is odd} \end{cases}$$

$$b_1 = \frac{1}{2}, \ b_n = 0, \ n = 2, 3, 4, \ldots$$

Hence FS

$$f(x) \approx \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + \frac{\cos 6x}{5\cdot 7} + \cdots\right) + \frac{1}{2}\sin x$$

Since series will converge to $f(x)$ for $x = \frac{\pi}{2}$ so

$$1 = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos \pi}{1\cdot 3} + \frac{\cos 2\pi}{3\cdot 5} + \frac{\cos 3\pi}{5\cdot 7} + \cdots\right) + \frac{1}{2}\sin\left(\frac{\pi}{2}\right)$$

$\Longrightarrow$

$$\frac{1}{4}(\pi - 2) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

3. Obtain a Fourier expression for $\sqrt{1 - \cos x}$ in the interval $-\pi < x < \pi$.

Solution: Hint:

$$f(x) = \sqrt{1 - \cos x} = \sqrt{2}\sin\left(\frac{x}{2}\right)$$

Now try to find the fourier series for $\sqrt{2}\sin\left(\frac{x}{2}\right)$.

4. Find the Fourier series for the function

$$f(x) = \begin{cases} -h & -\pi < x < 0 \\ h & 0 < x < \pi \end{cases}$$

here, $h \neq 0$. Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Solution: f is an odd function thus $A_n = 0$ for all n.

$$B_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4h}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Thus, the Fourier series is given by

$$f(x) \approx \frac{4h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$$

Since series will converge to $f(x)$ for $x = \dfrac{\pi}{2}$ so

$$f\left(\frac{\pi}{2}\right) = h = \frac{4h}{\pi}\left(\frac{\sin\left(\frac{\pi}{2}\right)}{1} + \frac{\sin\left(\frac{3\pi}{2}\right)}{3} + \frac{\sin\left(\frac{5\pi}{2}\right)}{5} + \ldots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

5. Determine the half-range Fourier sine series for $f(x) = x(\pi - x)$ in $0 < x < \pi$. Hence deduce

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

Solution: In order to find half-range Fourier sine series for $f(x)$ we need to do odd-periodic extension of $f(x)$

$$f_o(x) = \begin{cases} x(\pi - x) & 0 < x < \pi \\ x(\pi + x) & -\pi < x < 0 \\ f_o(x + 2\pi) & \text{otherwise} \end{cases}$$

Now we will find the FS for this new function $f_o(x)$

$$a_n = 0, n = 0, 1, 2, \ldots (\text{ since } f_o(x) \text{ is odd function })$$

$$b_n = \begin{cases} \dfrac{8}{\pi n^3} & \text{n is odd} \\ 0 & \text{n is even} \end{cases}$$

Hence FS for $f_o(x)$ in $-\pi < x < \pi$ is

$$f_o(x) \approx \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}$$

Since series will converge to $f_0(x)$ for $x = \dfrac{\pi}{2}$ so

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n-1)\pi}{2}\right)}{(2n-1)^3}$$

$$\implies \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$$

6. Determine the half-range Fourier sine series for $f(x) = e^x$ in $0 < x < \pi$.

Solution: Hint : Odd-periodic extension of $f(x)$

$$f_o(x) = \begin{cases} e^x & 0 < x < \pi \\ -e^{-x} & -\pi < x < 0 \\ f_o(x + 2\pi) & \text{otherwise} \end{cases}$$

Now find the fourier series for this odd-periodic extension

7. Determine the half-range Fourier cosine series for $f(x) = \sin(kx)$ for some k which is not an integer.

Solution: Let $f(x)$ be a function defind on $0 < x < L$.
Even-periodic extension of $f(x)$

$$f_e(x) = \begin{cases} \sin(kx) & 0 < x < L \\ -\sin(kx) & -L < x < 0 \\ f_e(x + 2L) & \text{otherwise} \end{cases}$$

Clearly $b_n = 0$ for $n = 1, 2, 3, \ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_{0}^{L}\sin(kx)\cos\left(\frac{n\pi x}{L}\right)dx$$

which after integrating gives

$$a_n = \frac{2kL}{(kL + n\pi)(kL - n\pi)}[1 - \cos(kL)]$$

The required fourier series for $f(x) = \sin(kx)$ in $0 < x < L$ is given by

$$\frac{1}{L}\left[\frac{1 - \cos(kL)}{k}\right] + \sum_{n=1}^{\infty}\frac{2kL}{(kL + n\pi)(kL - n\pi)}[1 - \cos(kL)]\cos\left(\frac{n\pi x}{L}\right)$$

8. Obtain the half range Fourier cosine series for

$$\begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases}$$

Solution: Hint : Even-periodic extension of $f(x)$

$$f_e(x) = \begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \\ \pi + x & -\pi < x < -\frac{\pi}{2} \\ 0 & -\frac{\pi}{2} < x < 0 \\ f_e(x + 2\pi) & \text{otherwise} \end{cases}$$

Now find the Fourier series for $f_e(x)$.

9. Form a partial differential equation by the method of elimination of arbitrary constants from the following

a) $z = a(x + y) + b(x - y) + abt + c$ where z, x, y and t are variables and others are constant.
b) $z = ax^3 + by^3$.
c) $\log(az - 1) = x + ay + b$.
d) $\frac{x}{a} + \frac{y}{b} - \frac{z}{ab} = 0$.

10. Form a partial differential equation by the method of elimination of arbitrary function from the following

a) $xyz = f(x + y + z)$
b) $z = f(x^2 - y^2)$
c) $F(x + y + z, x^2 + y^2 + z^2) = 0$
d) $F(ax + by + cz, x^2 + y^2 + z^2) = 0$

11. Solve the partial diferential equation

a) $(x^2 + y^2 + z^2)p - 2xyq = -2xz$
b) $pz - qz = z^2 + (x + y)^2$
c) $(y + z)p + (z + x)q = x + y$
d) $(mz - ny)p + (nx - lz)q = ly - mx$
e) $p + 3q = 5z + \tan(y - 3x)$

12. Find the temperature in a bar of length 2 whose ends kept at zero and lateral surface insulated if the initial temperature is $\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$.

13. A tightly stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibriu position. If it is set vibrating by giving to each of its points an initial velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$ then find the displacement $y(x, t)$ at any point of string at any time t.

14. The temperature distribution in a bar of length π, which is perfectly insulated at the ends $x = 0$ and $x = \pi$ is governed by the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. Assuming the initial temperature as $u(x, 0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time.

15. Solve the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, representing the vibration of a string of length l, fixed at both ends, given that $y(0, t) = 0, y(1, t) = 0, y(x, 0) = f(x)$ and $\frac{\partial}{\partial t} y(x, 0) = 0,\ 0 < x < l$.