DEPARTMENT OF MATHEMATICS

BIRLA INSTITUTE OF TECHNOLOGY MESRA, RANCHI

MA107 Mathematics-II, Session: (SP-2020)

Tutorial - 2 (Module II)

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1. Determine the singular points of the following differential equations and examine whether they are regular or not

   a) $(x^2+4)(x-3)^3y'' + (x-3)^3y' + y = 0$

   b) $y'' + \frac{1}{x}y' + \left(1 - \frac{1}{x^2}\right)y = 0$

   c) $x^2y'' + y' + 3xy = 0$

   d) $x^2y'' + xy' + (x^2 - p^2)y = 0$

   e) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

   f) $x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$

   g) $x^2(1-x^2)y'' + \frac{2}{x}y' + 4y = 0$

2. Consider the equation $y'' + xy' + y = 0$.

   a) Find its general solution $y = \sum a_n x^n$ in the form $y = a_0y_1(x) + a_1y_2(x)$, where $y_1(x)$ and $y_2(x)$ are power series.

   b) Use the ratio test to verify that the two series $y_1(x)$ and $y_2(x)$ converges for all x.

   c) Show that $y_1(x)$ is the series expansion of $e^{-x^2/2}$, use this fact to find the second independent solution.

3. Find the solution of the following equations in power series

   a) $xy' = 3(y+1)$

   b) $y'' - 3y' + 2y = 0$

   c) $2x(1-x)y'' + (1-x)y' + 3y = 0$

4. Use Frobenius method to find the solutions of the following equations

   a) $xy'' + y' + xy = 0$

   b) $xy'' + 3y' + 4x^3y = 0$

   c) $x(2+x^2)y'' - y' - 6xy = 0$

   d) $4xy'' + 2y' + y = 0$

   e) $x(1-x)y'' + 4y' + 2y = 0$

   f) $9x(1-x)y'' - 12y' + 4y = 0.$

5. Express the following expression in terms of Legendre polynomials

   a) $2 - x + 3x^2$

   b) $x^3 - 7x^2 - 28x - 15$

6. Prove that

   a) $P_n(1) = 1$

   b) $P_n(-1) = (-1)^n$

   c) $P_n'(1) = \frac{1}{2}n(n+1)$

   d) $P_n'(-1) = \frac{1}{2}(-1)^n[n(n+1)]$

   e) $P_n(-x) = (-1)^nP_n(x)$

7. Prove that $\int_{-1}^{1} P_n(x)(1-2xt+t^2)^{-1/2}dx = \frac{2t^n}{2n+1}$.

8. (**Recurrence relations for** $P_n(x)$) To show that

   1. $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x),\ n \geq 1$
   2. $nP_n(x) = xP_n^{'}(x) - P_{n-1}^{'}(x)$
   3. $(2n+1)P_n(x) = P_{n+1}^{'}(x) - P_{n-1}^{'}(x)$
   4. $(n+1)P_n(x) = P_{n+1}^{'}(x) - xP_n^{'}(x)$
   5. $(1-x^2)P_n^{'}(x) = n(P_{n-1}(x) - xP_n(x))$
   6. $(1-x^2)P_n^{'}(x) = (n+1)(xP_n(x) - P_{n+1}(x))$
   7. $(2n+1)(x^2-1)P_n^{'}(x) = n(n+1)(P_{n+1}(x) - P_{n-1}(x))$ (**Beltrami's result**)

9. Show that when n is an integer $J_{-n}(x) = (-1)^n J_n(x)$

10. Prove that

    a) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos(x)$ b) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin(x)$ c) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}$

11. Write the general solution of the following equations (in terms of Bessel functions)

    a) $x^2y^{''} + xy^{'} + (x^2 - 25)y = 0$ b) $x^2y^{''} + xy^{'} + (x^2 - \dfrac{9}{16})y = 0$

12. (**Recurrence relations for** $J_n(x)$) To show that

    a) $\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$ b) $\dfrac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x)$

    c) $J_n^{'}(x) = J_{n-1}(x) - (n/x)J_n(x)$ d) $J_n^{'}(x) = (n/x)J_n(x) - J_{n+1}(x)$

    e) $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$ f) $2J_n^{'}(x) = J_{n-1}(x) - J_{n+1}(x)$

13. Evaluate $\int x^3 J_3(x)dx$

Solution: Hint : from Recurrence relation $12(a)$ and $12(b)$

$$\int x^n J_{n-1}(x)dx = x^n J_n(x) + c$$

$$\int x^{-n} J_{n+1}(x)dx = -x^{-n} J_n(x) + c$$