What is optimization?

Optimization is essentially about finding the best solution to a given problem from a set of feasible solutions. It consist of three components

- the objective or objectives, that is, what do we want to optimize?
- a solution (decision) vector, that is, how can we achieve the optimal objective?
- the set of all feasible solutions, that is, among which possible option may we choose to optimize?

Examples

- Airline companies schedule crews and aircraft to minimize their cost
- Investors create portfolios to avoid the risks and achieve the maximum profits
- Manufactures minimize the production costs and maximize the efficiency
- Bidders optimize their bidding strategies to achieve best results
- Physical system tends to a state of minimum energy

What are necessary and sufficient conditions for a local minimum?

- Necessary conditions: conditions satisfied by every local minimum
- Sufficient conditions: conditions which guarantee a local minimum

Easy to characterize a local minimum if f is sufficiently smooth

Formulation of optimization Problems: Degree of freedom, Constraints, Objective function, Continuity of function, unimodal function and multimodal func. Concave and convex function

Stationary points

Let $f: \mathbb{R} \to \mathbb{R}$, $f \in C^1$

Consider the problem, $\min_{x \in \mathbb{R}} f(x)$

Definition

x^* is called a stationary point if $f'(x^*) = 0$

Necessity of an algorithm

- Consider the problem

$$\min_{x \in \mathbb{R}} (x-2)^2$$

we first find the stationary points (which satisfy $f'(x) = 0$)

$$f'(x) = 0 \Rightarrow 2(x-2) = 0 \Rightarrow x^* = 2$$

$f''(2) = 2 > 0 \Rightarrow x^*$ is a strict local minimum

- Stationary points are found by solving a nonlinear equation

$$g(x) = f'(x) = 0$$

Finding the real roots of $g(x)$ may not be always easy

- consider the problem to minimize $f(x) = x^2 + e^x$
- $g(x) = 2x + e^x$
- Need an algorithm to find x which satisfies $g(x) = 0$

One dimensional optimization
- Derivative free method (search methods)
- Derivative based method (Approximation methods)
- Inexact methods

Degree of Freedom

To detemine the degrees of freedom (the number of variables whose values may be independently speified) in our model we could simply count the number of independent variables (the number of variables which remain on the right-hand side in our modefied equations.

This suggests a possible definition

degree of freedom = variables - equations

The degree of freedom for a given problem are the number of independent problem variables which must be specified to uniquely determine a solution

· Production allocation Problem

Four different metals namely iron, zinc, copper and manganese are required to produce three comodeties A, B and C. To produce one unit of A 40 kg iron, 30 kg copper, 7 kg zinc and 4 kg manganese are needed. Similarly to produce one unit of B 70 kg iron, 14 kg copper and 9 kg manganese are needed and for producing one unit of C 50 kg iron, 18 kg copper and 8 kg zinc are required. The total available quantities of metals are 1 metric ton iron, 5 quintals of copper 2 quintals of zinc and manganese each. The profits are Rs-300, Rs-200 and Rs-100 in selling per one unit of A, B and C respectively. Formulate the problem mathematically

Sol Let Z be the total profit and the problem is to maximize Z

Z is known as the objective function

All the available quantities and the quantities required to produce different commodities are given below in the tabular form

	Iron	Copper	Zinc	Manganese
Total	1000	500 kg	200 kg	200 kg
A	40 kg	30 kg	7 kg	4 kg
B	70 kg	14 kg	0 kg	9 kg
C	50 kg	18 kg	8 kg	0 kg

To get the maximum profit let x_1 unit of A, x_2 units of B and x_3 unit of C are to be produced

$$40x_1 + 70x_2 + 50x_3 \leq 1000$$
$$30x_1 + 14x_2 + 18x_3 \leq 500$$
$$7x_1 + 0x_2 + 8x_3 \leq 200$$
$$4x_1 + 9x_2 + 0x_3 \leq 200$$

max $Z = 300x_1 + 200x_2 + 100x_3$

∴ objective function

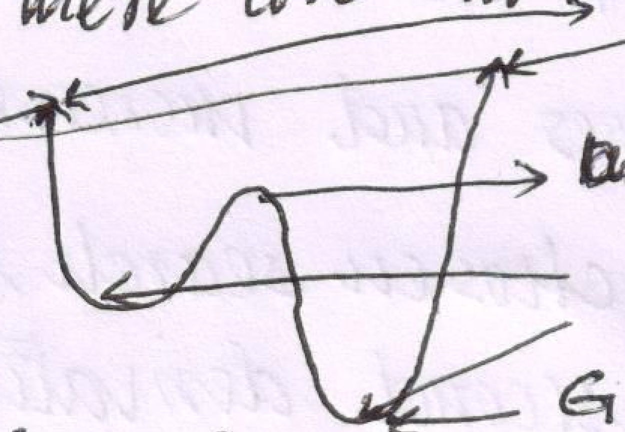
None of the commodity produced be negative in number then $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

All these inequations are known as constraints or restrictions.

Unbounded Optimization

Optimization means you are trying to find a maximum or minimum value

Unbounded means there are no constraints on the function - it keeps going forever



Local vs. Global extremes

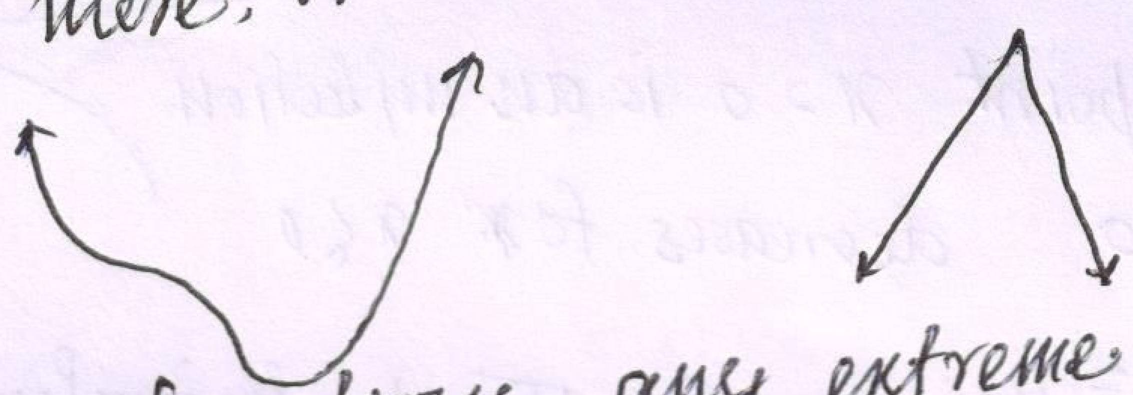
If a point is a maximum or minimum relative to all the other points on the function then it is consider a global maximum or global minimum

1 Sketch a function with
   a) Two local maxima, one of which is global, one local minimum and no global minimum
   b) No local or global extremes
   c) One global minimum and no maxima
   d) Two global minima, one local maximum, no global maximum.

2. Write a set of conditions that would be impossible

Unimodal and Multimodal Functions

A unimodal function has only one minimum and the rest of the graph goes up from there, or one maximum and the rest of the graph goes down



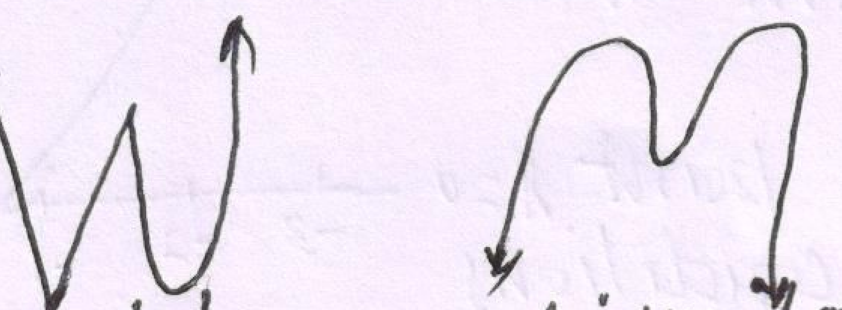
Let $\phi : \mathbb{R} \to \mathbb{R}$, consider the problem $\min_{x \in \mathbb{R}} \phi(x)$, Let x^+ be the minimum point and $x^+ \in [a, b]$. The function ϕ is said to be unimodal on $[a, b]$ if for $a \leq x_1 < x_2 \leq b$

$x_2 < x^+ \Rightarrow \phi(x_1) > \phi(x_2)$

$x_1 > x^+ \Rightarrow \phi(x_2) > \phi(x_1)$.

with unimodal functions any extreme you find is guaranteed to be the global extreme

A bimodal function has two local minima or maxima



Beyond that, trimodal, quadrimodal and then multimodal

with bimodal and above, you don't know if an extreme is local or global unless you know the entire graph

Single variable optimization algorithms

The algorithms described in this section can be used to solve minimization problems of the following type

Minimize $f(x)$ - where $f(x)$ is the objective function and x is a real variable.

The purpose of an optimization algorithm is to find a solution x, for which the function is minimum

- Optimality criteria -

There are three different types of optimal points are

(i) Local optimal point

A point or solution x^* is used to be a local optimal point, if no point in the neighbourhood has a function value smaller than $f(x^*)$

(ii) Global Optimal point

A point or solution x^{**} is said to be a global optimal point, if no point in the entire search space has a function value smaller than $f(x^{**})$

(iii) Inflection point

x^* is an inflection point if

$f(x^*)$ increases locally as x^* increases & decreases locally as x^* reduces

or $f(x^*)$ decreases locally as x^* increases and increases locally as x^* decreases

Let the objective function $f(x)$ is the chosen search space

$f'(x)$ and $f''(x)$ are first and second derivatives

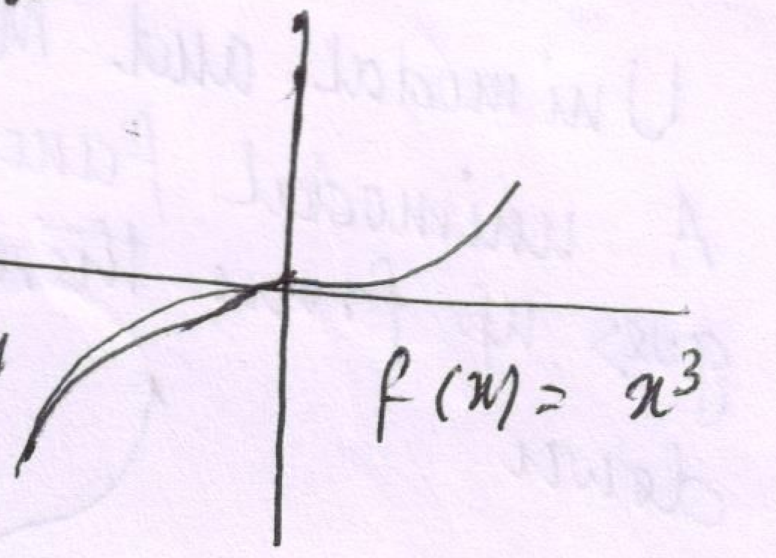
A point x is a minimum if $f'(x) = 0$ & $f''(x) \geq 0$

if $f'(x) = 0$ the point is either a minimum, a maximum or an inflection point

Suppose that at point x^*, the first derivative is zero and the first non-zero higher order derivative is denoted by 'n' then

- If n is odd x^* is an inflection point
- If n is even x^* is a local optimum
  - (i) If the derivative is +ve, x^* is a local minimum
  - (ii) If the derivative is −ve, x^* is a local maximum

Example:- Consider $f(x) = x^3$ optimal point $x = 0$ as



From the figure we can see that point $x = 0$ is an inflection point as $f(x)$ increases for $x \geq 0$ decreases for $x \leq 0$

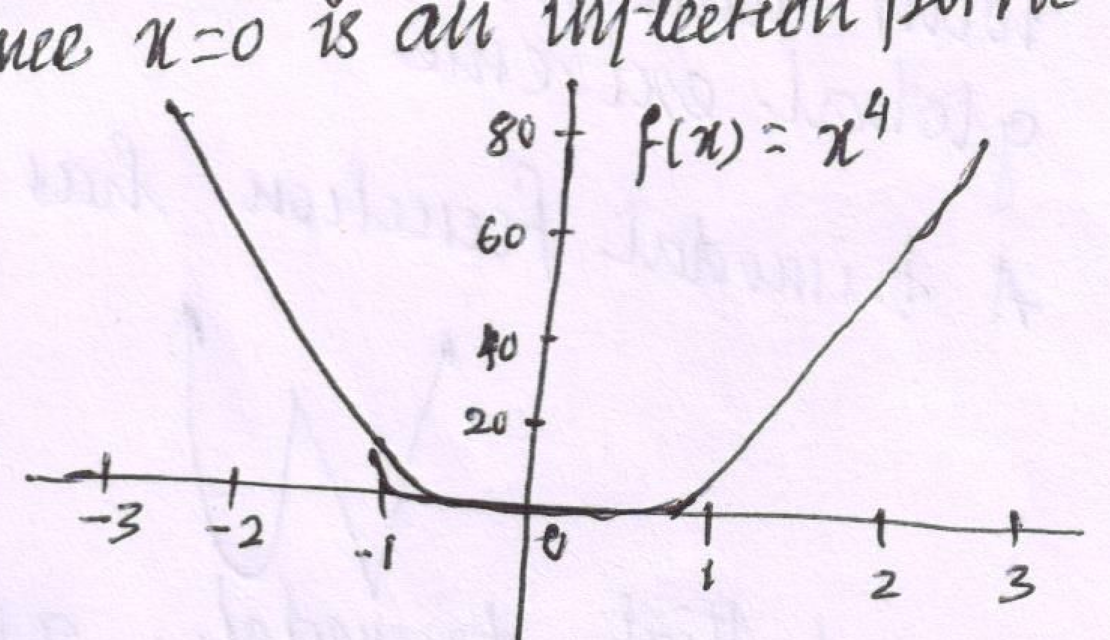
$f'(x=0) = 3x^2|_{x=0} = 0$

$f''(x=0) = 6x|_{x=0} = 0$

$f'''(x=0) = 6$ (Non zero value)

Third derivative $n = 3$ is odd hence $x = 0$ is an inflection point

Exmp: Consider $f(x) = x^4$, optimal point $x = 0$



From the figure, we can see that the point $x = 0$ is a minimal point. Using sufficient conditions

$f'(x=0) = 4x^3|_{x=0} = 0$

$f''(x=0) = 12x^2|_{x=0} = 0$

$f'''(x=0) = 24x|_{x=0} = 0$

$f''''(x=0) = 24|_{x=0} = 24$ (Nonzero value)

Fourth order derivative is positive, $n = 4$ is even, hence $x = 0$ is a local minimum point.

Find the value of c so that the function $f(x) = \begin{cases} 2x + c & \text{if } x \leq 3 \\ 2c - x & \text{if } x > 3 \end{cases}$ is continuous.

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$

$6 + c = 2c - 3$

$\underline{\underline{c = 9}}$

If a is in the domain of $f(x)$ that is $f(a)$ exists as a number

$\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$