CL4001 HEAT TRANSFER OPERATIONS

MODULE

II

UNSTEADY STATE CONDUCTION

Lecture Notes:
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UNSTEADY STATE HEAT CONDUCTION

- **The temperature of a body, in general, varies with time as well** as position. In rectangular coordinates, this variation is expressed as *T(x, y, z, t), where (x, y, z) indicates variation in the x, y, and z directions*, respectively, and *t indicates variation with time.*
- *In the preceding chapter, we* considered heat conduction under *steady conditions*, *for which the temperature* of a body at any point does not change with time.
- This certainly simplified the analysis, especially when the temperature varied in one direction only.

- Unsteady-state heat conduction

$$\frac{\delta^2 T}{\delta x^2} = \frac{1}{\alpha}.\frac{\delta T}{\delta \tau}$$

$$\Rightarrow \frac{\delta^2 \theta}{\delta x^2} = \frac{1}{\alpha}.\frac{\delta \theta}{\delta \tau}$$

- Boundary conditions

$$\theta = \theta_i = T_i - T_1 \quad \text{at } \tau = 0,\ 0 \le x \le 2L$$

$$\theta = 0 \quad \text{at } x = 0,\ \tau > 0$$

$$\theta = 0 \quad \text{at } x = 2L,\ \tau > 0$$

- Solution:

$$\theta = X(x)\mathrm{H}(\tau)$$

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0$$

$$\frac{d^2 H}{d\tau^2} - \alpha\lambda^2 H = 0$$

- Final solution for temperature distribution:

$$\frac{T - T_1}{T_i - T_1} = \frac{4}{\Pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp(-\sin(\Pi x / W) \frac{\sinh(\Pi y / W)}{\sinh(\Pi H / W)}$$

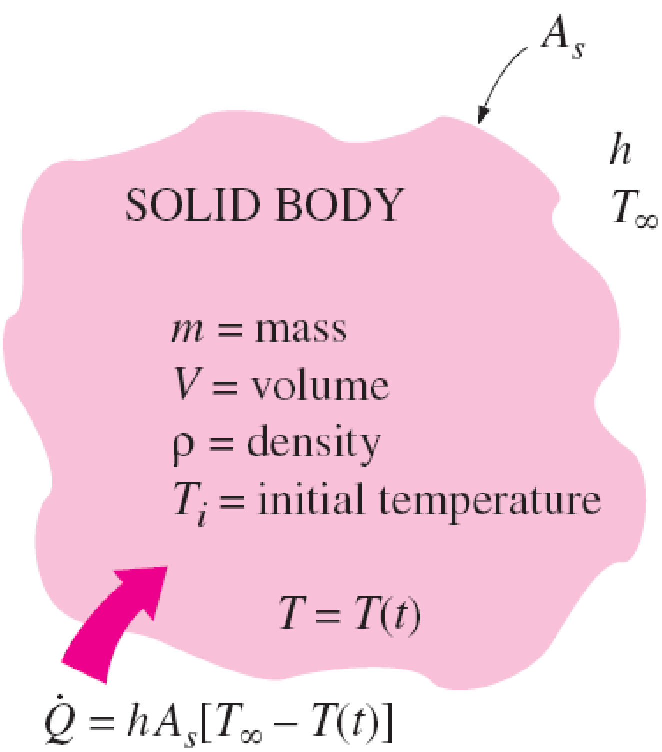
UNSTEADY STATE HEAT CONDUCTION (lumped systems)

- In heat transfer analysis, some bodies are observed to behave like a "lump" whose interior temperature remains essentially uniform at all times during a heat transfer process. The temperature of such bodies can be taken to be a function of time only, *T(t)*. *Heat transfer analysis that utilizes this idealization* is known as **lumped system analysis, which provides great simplification** in certain classes of heat transfer problems without much sacrifice from accuracy.
- Consider a small hot copper ball coming out of an oven. Measurements indicate that the temperature of the copper ball changes with time, but it does not change much with position at any given time. Thus the temperature of the ball remains uniform at all times, and we can talk about the temperature of the ball with no reference to a specific location.

UNSTEADY STATE HEAT CONDUCTION (lumped systems)

- Example: consider a large roast in an oven. If you have done any roasting, you must have noticed that the temperature distribution within the roast is not even close to being uniform. You can easily verify this by taking the roast out before it is completely done and cutting it in half. You will see that the outer parts of the roast are well done while the center part is barely warm. Thus, lumped system analysis is not applicable in this case. Before presenting a criterion about applicability of lumped system analysis, we develop the formulation associated with it.

UNSTEADY STATE HEAT CONDUCTION (lumped systems)



The geometry and parameters involved in the lumped system analysis.

Consider a body of arbitrary shape of mass *m*, *volume V, surface area* A_s, density , and specific heat C_p *initially at a uniform temperature* T_i *(Fig.).*

At time *t = 0, the body is placed into a medium at temperature* T_∞*, and heat* transfer takes place between the body and its environment, with a heat transfer coefficient *h. For the sake of discussion, we will assume that* $T_\infty > T_i$*, but* the analysis is equally valid for the opposite case. We assume lumped system analysis to be applicable, so that the temperature remains uniform within the body at all times and changes with time only, $T = T(t)$.

UNSTEADY STATE HEAT CONDUCTION (lumped systems)

- During a differential time interval *dt, the temperature of the body rises by a* differential amount *dT. An energy balance of the solid for the time interval dt* can be expressed as

$$\begin{pmatrix}\text{Heat transfer into the body} \\ \text{during } dt\end{pmatrix} = \begin{pmatrix}\text{The increase in the} \\ \text{energy of the body} \\ \text{during } dt\end{pmatrix}$$

$$hA_s(T_\infty - T)\,dt = mC_p\,dT \qquad (2.1)$$

Noting that $m = \rho V$ *and* $dT = d(T - T_\infty)$ *since* T_∞ *constant, Eq. (2–1) can be* rearranged as

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho V C_p}\,dt \qquad (2.2)$$

UNSTEADY STATE HEAT CONDUCTION (lumped systems)

- Integrating from *t = 0, at which T = T_i, to any time t, at which T = T(t), gives*

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho V C_p} t \qquad (2.3)$$

or

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} \qquad (2.4)$$

where

$$b = \frac{hA_s}{\rho V C_p} \qquad (2.5)$$

UNSTEADY STATE HEAT CONDUCTION (lumped systems)

- b is a positive quantity whose dimension is $(\text{time})^{-1}$. The reciprocal of *b has* time unit (usually s), and is called the **time constant.**
- The temperature of a body approaches the ambient temperature *T* exponentially. The temperature of the body changes rapidly at the beginning, but rather slowly later on.
- A large value of *b indicates that* the body will approach the environment temperature in a short time.
- The larger the value of the exponent *b, the higher the rate of decay in* temperature.
- *b is proportional to the surface area, but inversely* proportional to the mass and the specific heat of the body. This is not surprising since it takes longer to heat or cool a larger mass, especially when it has a large specific heat.

- **Criteria for Lumped System Analysis**

The lumped system analysis certainly provides great convenience in heat transfer analysis. The first step in establishing a criterion for the applicability of the lumped system analysis is to define a **characteristic length as**

$$L_c = \frac{V}{A_s} \qquad (2.6)$$

And ***Biot No***. as

$$\mathrm{Bi} = \frac{hL_c}{k} \qquad (2.7)$$

$$\mathrm{Bi} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

$$\mathrm{Bi} = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$

UNSTEADY STATE HEAT CONDUCTION (lumped systems)

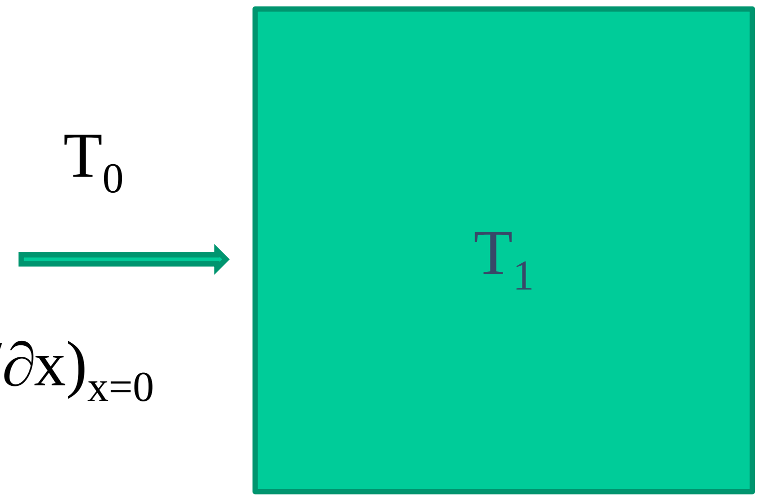
Significance of Biot No. :

- *The Biot number is the ratio of the* internal resistance of a body to *heat conduction to its external resistance to heat convection.*
- *a small Biot number represents small resistance* to heat conduction, and thus small temperature gradients within the body.
- Lumped system analysis assumes a *uniform temperature distribution* throughout the body, which will be the case only when the thermal resistance of the body to heat conduction (the *conduction resistance) is zero*.
- *Thus*, lumped system analysis is *exact when Bi = 0 and approximate*
- *when Bi >0.* Of course, the smaller the Bi number, the more accurate the lumped system analysis.
- It is generally accepted that lumped system analysis is *applicable if* $Bi \leq 0.1$ When this criterion is satisfied, the temperatures within the body relative to the surroundings (i.e., $T - T_\infty$) *remain within 5 percent of each other even for* well-rounded geometries such as a spherical ball.

TRANSIENT HEAT FLOW IN A SEMI-INFINITE SOLID

Let us consider a semi-infinite solid as shown in figure which is maintained at some initial temperature T_i. The surface temperature is suddenly lowered and maintained at a temp $T_{0,}$ and the expression for the temp distribution in the solid as a function of time and position will be developed.

The differential equation for the temp distribution is



$q_0 = -kA(\partial T/\partial x)_{x=0}$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\delta T}{\delta \tau} \qquad (2.8)$$

TRANSIENT HEAT FLOW IN A SEMI-INFINITE SOLID

- The boundary conditions to solve the above differential equation are
- $T(x,0) = T_i$ and $T(0,\tau) = T_0$ for $\tau > 0$ (2.9)
- The differential equation will be solved using laplace-technique and the solution is as follows

$$\frac{T(x,\tau)-T_0}{T_i-T_0} = \frac{2}{\sqrt{\pi}} \int^{x/2\sqrt{\alpha\tau}} e^{-\eta^2} d\eta \quad (2.10)$$

- Where η is the dummy variable.

Therefore heat flow at any position x may be obtained from

$$q_x = -kA\frac{\delta T}{\delta x}$$

From differential eq. (2.10)

$$\frac{\delta T}{\delta x} = (T_i - T_0)\frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4\alpha\tau}}\frac{\partial}{\partial x}\left(\frac{x}{2\sqrt{\alpha\tau}}\right) \qquad (2.11)$$

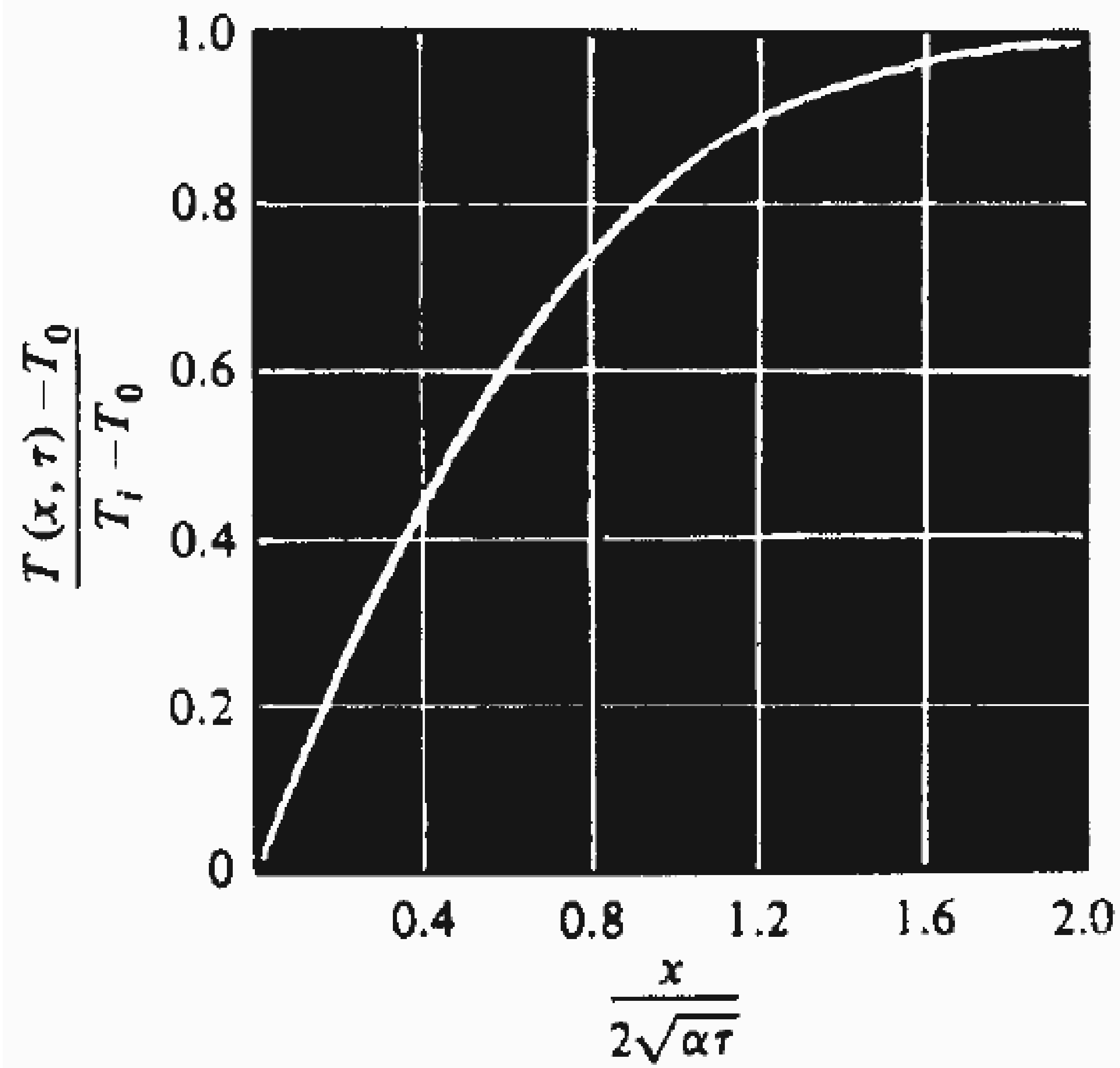
$$= \frac{(T_i - T_0)}{\sqrt{\pi\alpha\tau}}e^{-\frac{x^2}{4\alpha\tau}}$$

At the surface x=0 the heat flow is

$$q_0 = \frac{kA(T_i - T_0)}{\sqrt{\pi\alpha\tau}} \qquad (2.12)$$

TRANSIENT HEAT FLOW IN A SEMI-INFINITE SOLID

The surface heat flux is determined by evaluating the temperature gradient at x=0 from above eq. a plot of the temperature distribution for semi-infinite solid is given by the figure.

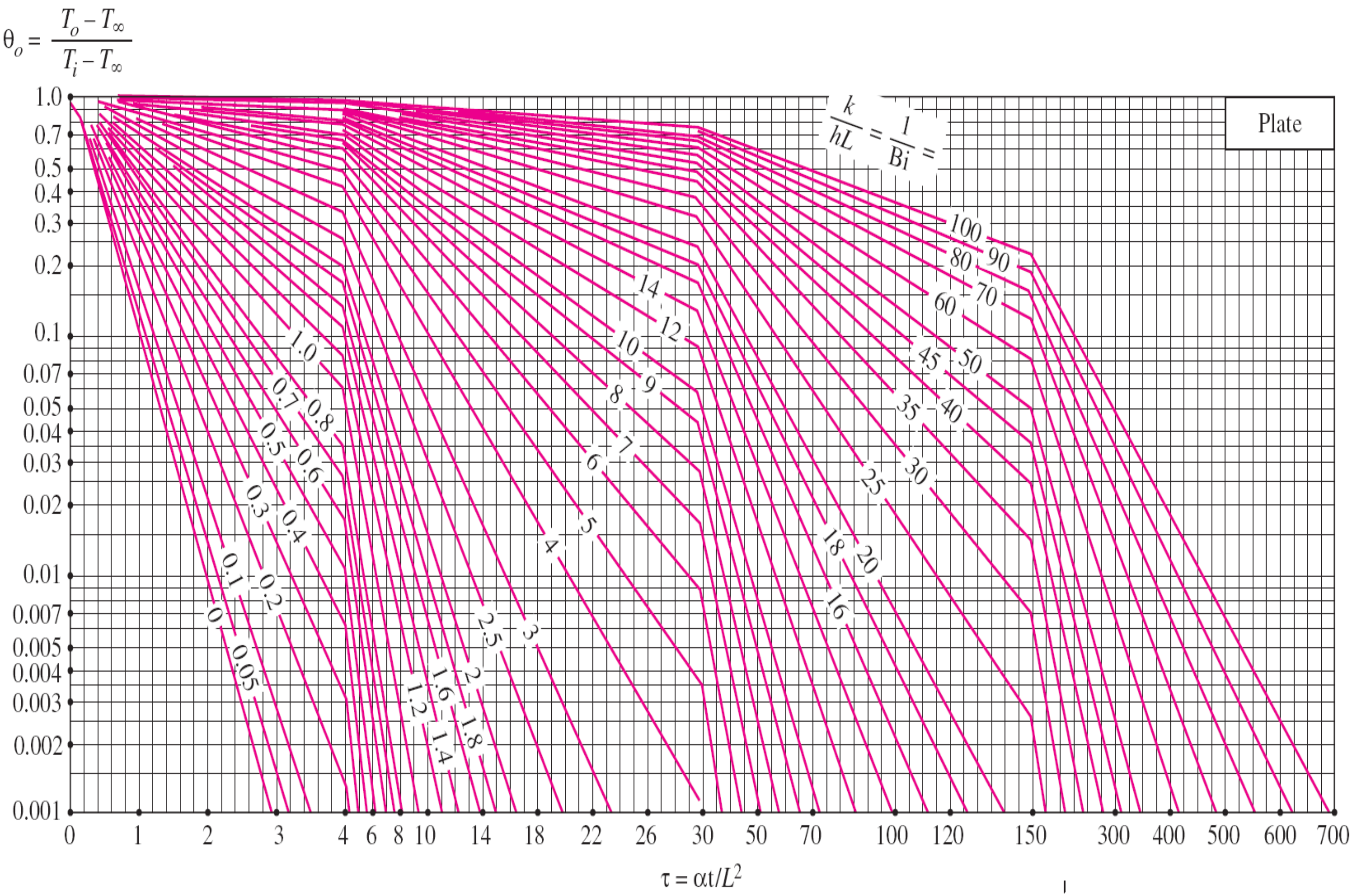


CONVECTION AND HEAT TRANSFER COEFFICIENT

- The rate of heat transfer in forced convection depends on properties of both the fluid (density, heat capacity, etc.) and of the flow (geometry, turbulence, etc.). The calculation is generally complex, and may involve boundary layer theory and tricky mathematics, so we typically use empirical correlations based on masses of data. These enable us to determine *heat transfer coefficients* for use in calculations.
- A ***heat transfer coefficient***, *h*, is the proportionality factor between the heat flux and an overall temperature difference driving force:

$$(q/A) = h\Delta T_{mean}$$

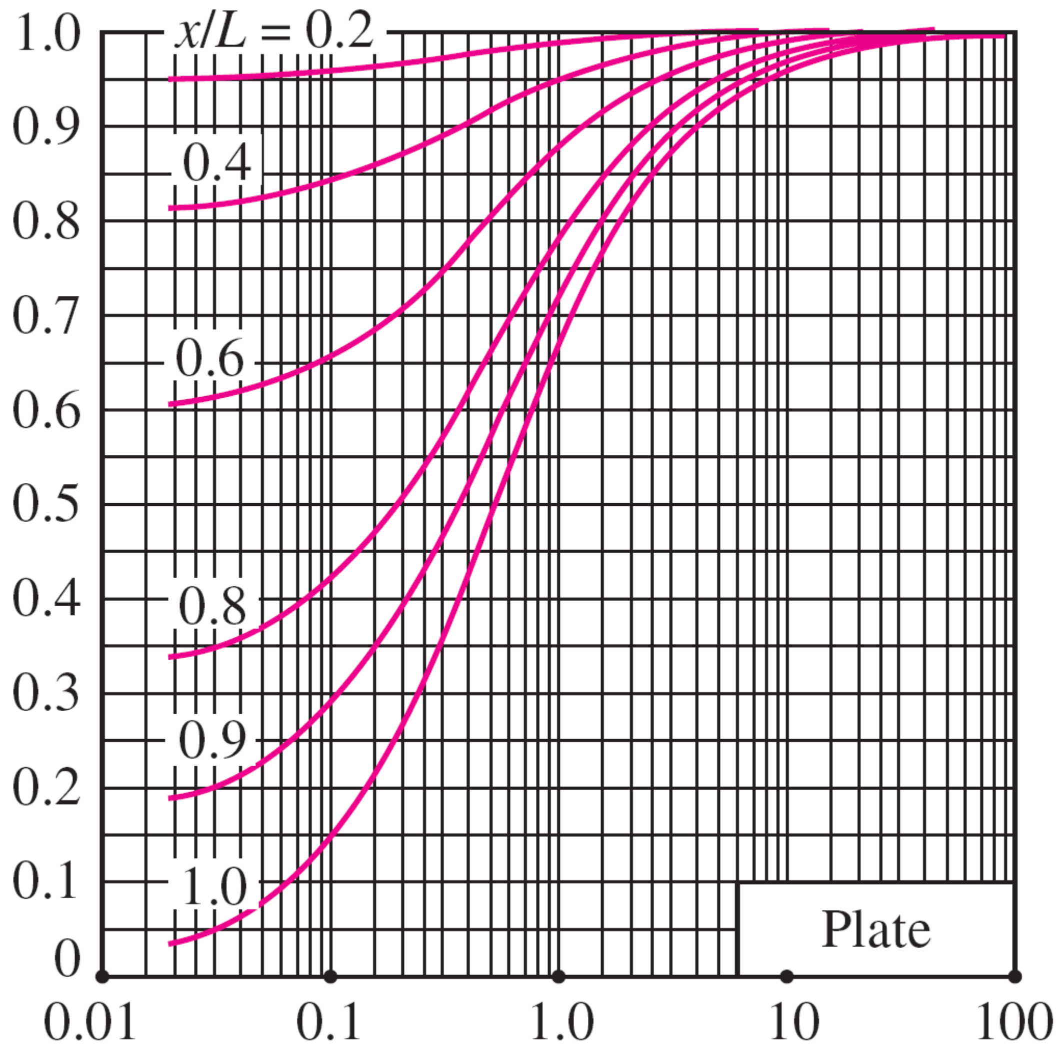
Graphical Representation of the One-Term Approximation: The *HeislerCharts*



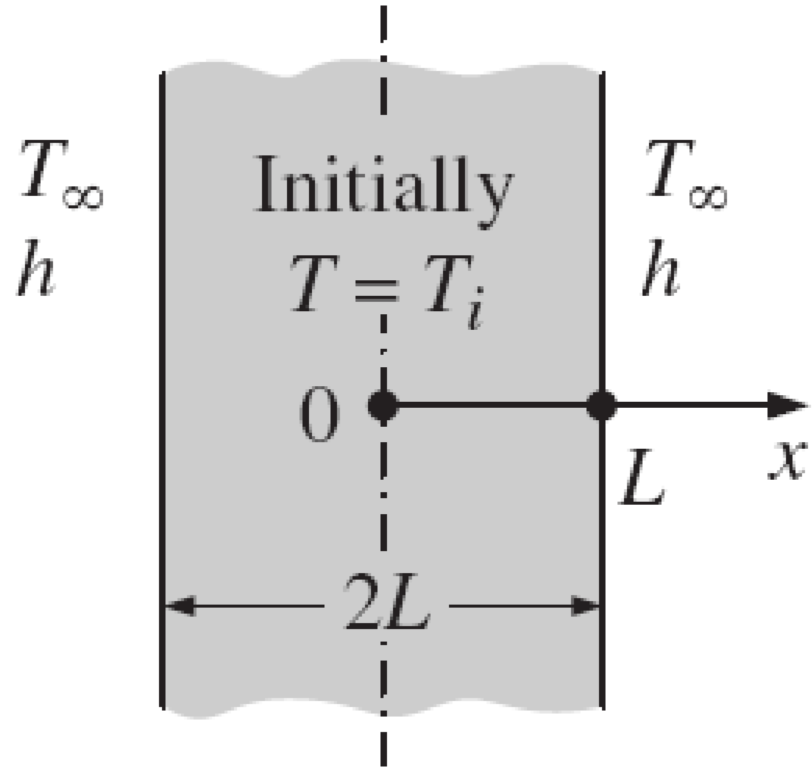
Midplane temperature (from M. P. Heisler)

Graphical Representation of the One-Term Approximation: The *HeislerCharts*

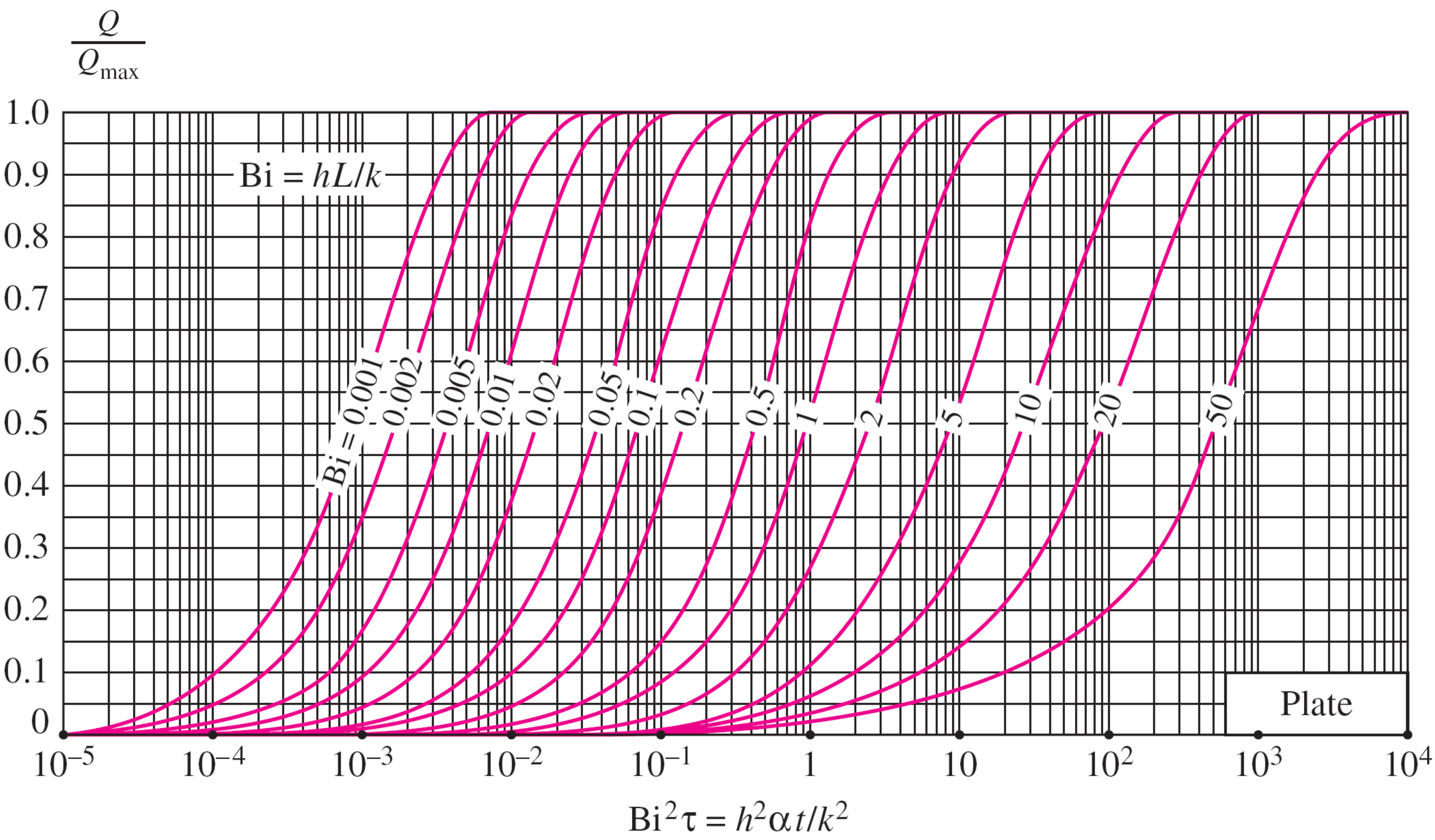
$$\theta = \frac{T - T_\infty}{T_o - T_\infty}$$



$$\frac{1}{\text{Bi}} = \frac{k}{hL}$$



Graphical Representation of the One-Term Approximation: The *HeislerCharts*



Numerical Methods for Unsteady Heat Transfer

- Unsteady heat transfer equation, no generation, constant k, two-dimensional in Cartesian coordinate:

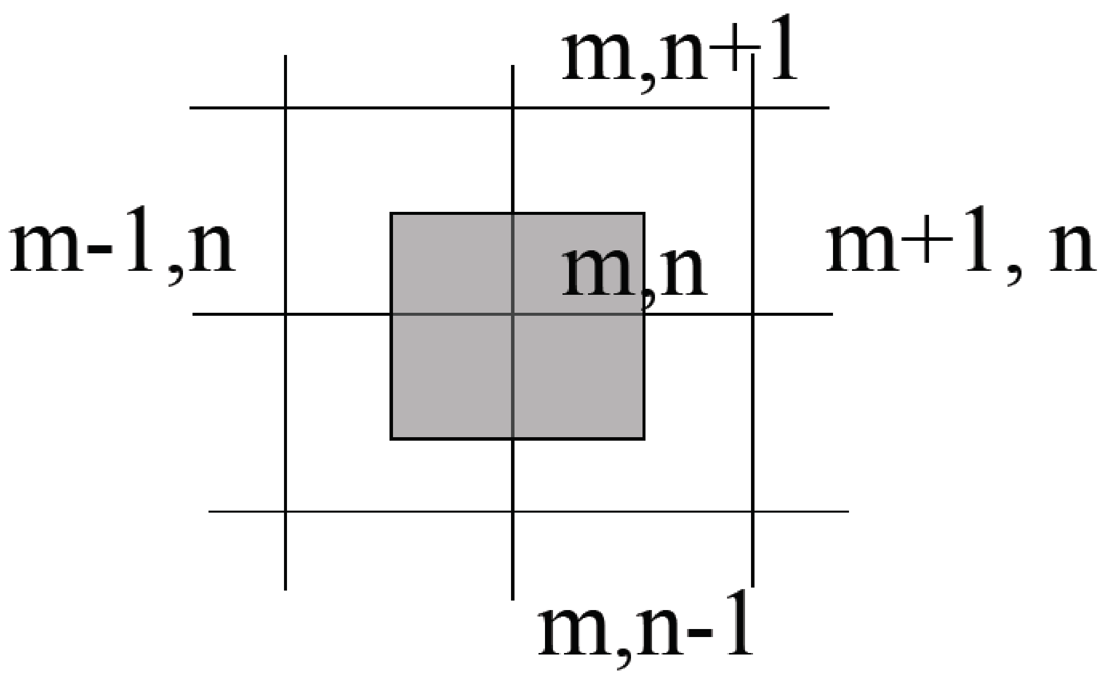
$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

- We have learned how to discretize the Laplacian operator into system of finite difference equations using nodal network. For the unsteady problem, the temperature variation with time needs to be discretized too. To be consistent with the notation from the book, we choose to analyze the time variation in small time increment Δt, such that the real time t=pΔt. The time differentiation can be approximated as:

$$\left.\frac{\partial T}{\partial t}\right|_{m,n} \approx \frac{T_{m,n}^{P+1} - T_{m,n}^{P}}{\Delta t},$$ while m & n correspond to nodal location

such that x=mΔx, and y=nΔy as introduced earlier.

Finite Difference Equations



From the nodal network to the left, the heat equation can be written in finite difference form:

Finite Difference Equations

$$\frac{1}{\alpha}\frac{T_{m,n}^{P+1}-T_{m,n}^{P}}{\Delta t}=\frac{T_{m+1,n}^{P}+T_{m-1,n}^{P}-2T_{m,n}^{P}}{(\Delta x)^2}+\frac{T_{m,n+1}^{P}+T_{m,n-1}^{P}-2T_{m,n}^{P}}{(\Delta y)^2}$$

Assume Δx=Δy and the discretized Fourier number $Fo=\dfrac{\alpha\Delta t}{(\Delta x)^2}$

$$T_{m,n}^{P+1}=Fo\left(T_{m+1,n}^{P}+T_{m-1,n}^{P}+T_{m,n+1}^{P}+T_{m,n-1}^{P}\right)+(1-4Fo)T_{m,n}^{P}$$

This is the **explicit**, finite difference equation for a 2-D, unsteady heat transfer equation.

The temperature at time p+1 is explicitly expressed as a function of neighboring temperatures at an earlier time p

- PROBLEM: The semi-infinite Aluminium slab of Example 4-4 is suddenly exposed to a convection-surface environment of 70◦C with a heat-transfer coefficient of 525 W/m2 · ◦C. Calculate the time required for the temperature to reach 120◦C at the depth of 4.0 cm for this circumstance.
- Solution:

$$hA(T_\infty - T)_{x=0} = -kA \left.\frac{\partial T}{\partial x}\right]_{x=0}$$

$$\frac{T - T_i}{T_\infty - T_i} = 1 - \operatorname{erf} X - \left[\exp\left(\frac{hx}{k} + \frac{h^2\alpha\tau}{k^2}\right)\right] \times \left[1 - \operatorname{erf}\left(X + \frac{h\sqrt{\alpha\tau}}{k}\right)\right]$$

where

$$X = x/(2\sqrt{\alpha\tau})$$

T_i = initial temperature of solid

T_∞ = environment temperature

- Aaa

$$\frac{T - T_i}{T_\infty - T_i} = \frac{120 - 200}{70 - 200} = 0.615$$

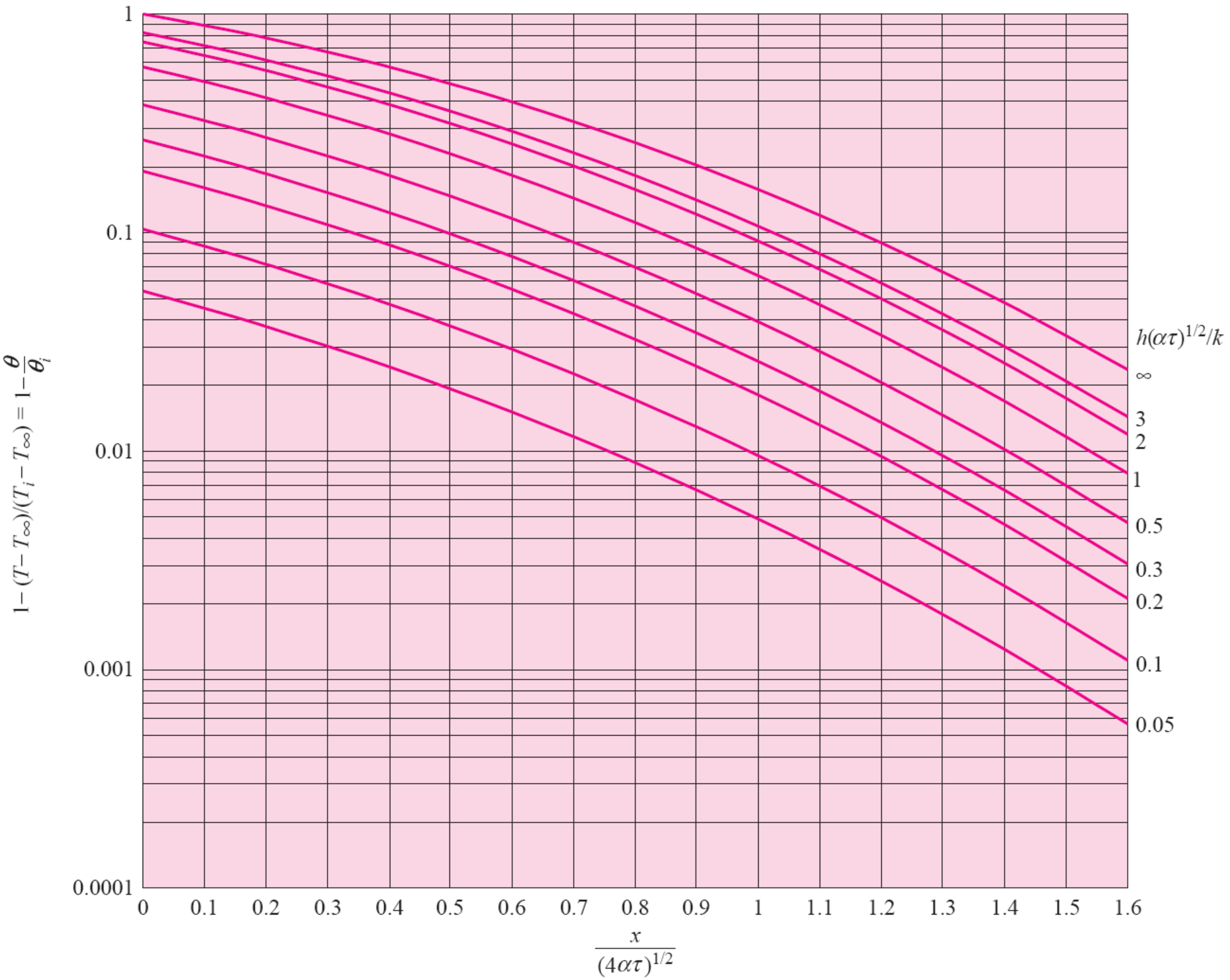
τ, s	$\frac{h\sqrt{\alpha\tau}}{k}$	$\frac{x}{2\sqrt{\alpha\tau}}$	$\frac{T - T_i}{T_\infty - T_i}$ **from Figure 4-5**
1000	0.708	0.069	0.41
3000	1.226	0.040	0.61
4000	1.416	0.035	0.68

- the time required is approximately 3000 s.

Aluminum Plate Suddenly Exposed to Convection

- PT\ROBLEM 2: A large plate of aluminum 5.0 cm thick and initially at 200◦C is suddenly exposed to the convection environment. Calculate the temperature at a depth of 1.25 cm from one of the faces 1 min after the plate has been exposed to the environment. How much energy has been removed per unit area from the plate in this time?
- Solution:

Temperature distribution in the semi-infinite solid with convection boundary condition. (ref: Holman fig 4.5)



Heisler charts
