HEAT TRANSFER OPERATIONS

MODULE

I

CONDUCTION

Lecture Notes:
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INTRODUCTION

- ***Heat Transfer is "Energy in transit due to temperature difference."***
- *Thermodynamics tells us:*
  - How much heat is transferred (δQ)
  - How much work is done (δW)
  - Final state of the system
- *Heat transfer tells us:*
  - How (with what modes) δQ is transferred
  - At what rate δQ is transferred
  - Temperature distribution inside the body

Heat transfer ⟷ complementary ⟷ Thermodynamics

INTRODUCTION

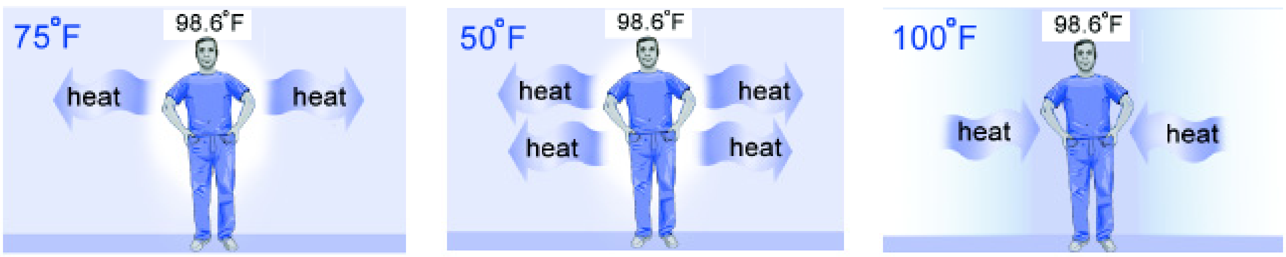
- ***What is the direction of Heat energy transfer from one substance to another?***
- ***If you put a hot cup of coffee into a refrigerator, would heat transfer from the fridge to the coffee, or would heat transfer from the coffee to the fridge?***

Hmmmm….

- As per 2nd law of thermodynamics, since the inside of refrigerator has a lower temperature than the coffee, heat energy travels from the coffee to the refrigerator, following the temperature gradient.
- ***Heat flow depends on the temperature difference.***

INTRODUCTION

- The science of how heat flows is called ***heat transfer***.
- There are three ways heat transfer works: ***conduction***, ***convection***, and ***radiation***.



- ***Conduction*** is the process whereby heat is transferred directly through a material, any bulk motion of the material playing no role in the transfer.
- ***Convection*** is when heat is carried by a moving fluid. Example: heat house with radiator, Gulf stream transports Heat from Caribbean to Europe
- ***Radiation*** is when electromagnetic waves (radiation) carry heat from one object to another. Example: heat you feel when you are near a fire, Heat from the sun, Formation of frost (ice) at night, T(air) > 0ºC

INTRODUCTION

- **Conduction**
  - needs matter
  - molecular phenomenon (diffusion process)
  - without bulk motion of matter

- **Convection**
  - heat carried away by bulk motion of fluid
  - needs fluid matter

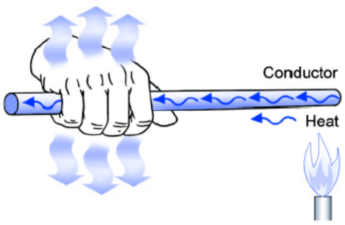
- **Radiation**
  - does not needs matter
  - transmission of energy by electromagnetic waves

APPLICATIONS OF HEAT TRANSFER

- Energy production and conversion
  - *steam power plant, solar energy conversion etc.*
- Refrigeration and air-conditioning
- Domestic applications-ovens, stoves, toaster
- Cooling of electronic equipment
- Manufacturing / materials processing
  - *welding, casting, soldering, laser machining*
- Automobiles / aircraft design
- Nature (weather, climate etc)

CONDUCTION

- **Heat is the form of energy that can be transferred** from one system to another as a result of temperature difference.
- The science that deals with the determination of the *rates of such energy transfers is the **heat*** **transfer.**
- The transfer of energy as heat is always from the higher-temperature medium to the lower-temperature one, and heat transfer stops when the two mediums reach the same temperature.
- **Conduction is the transfer of energy from the more energetic particles of a** substance to the adjacent less energetic ones as a result of interactions between the particles. On a molecular level, hotter molecules are vibrating faster than cooler ones. When they come in contact, the faster moving molecules "bump into" the slower moving molecules and heat is transferred.

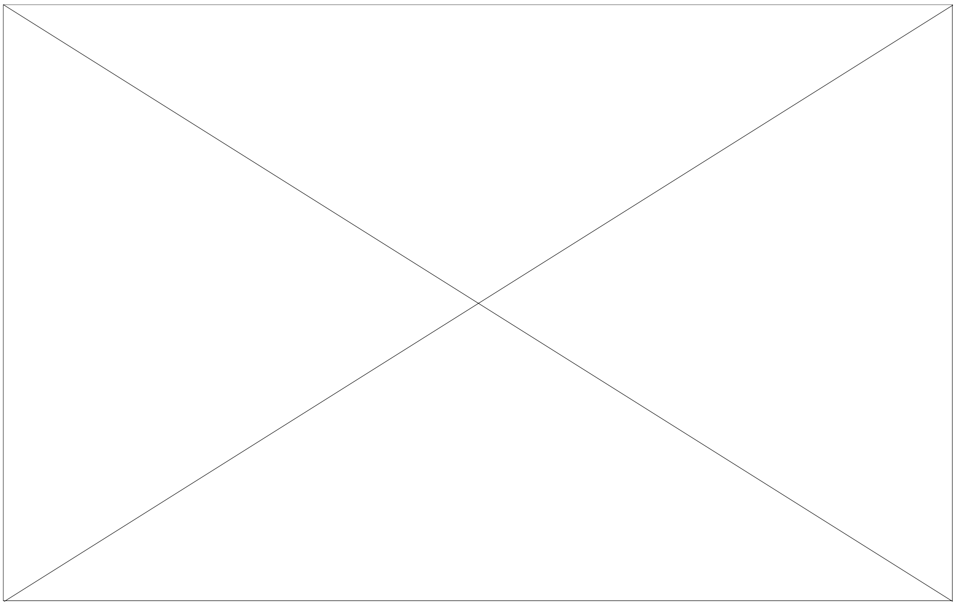


CONDUCTION

- Conduction can take place in solids, liquids, or gases. In gases and liquids, conduction is due to the *collisions and diffusion of the* molecules during their random motion. In solids, it is due to the combination of *vibrations of the molecules in a lattice and the energy transport by free electrons.*
- *Example: A cold canned drink in a warm room, for example, eventually* warms up to the room temperature as a result of heat transfer from the room to the drink through the aluminum can by conduction.
- The *rate of heat conduction through a medium depends on the geometry of* the medium, its *thickness, and the material of the medium, as well as the temperature difference across the medium. The wrapping a hot water* tank with glass wool (an insulating material) reduces the rate of heat loss from the tank. The thicker the insulation, the smaller the heat loss.
- ***The ability to conduct heat often depends more on the structure of a material than on the material itself.***

CONDUCTION

When you heat a metal strip at one end, the heat travels to the other end.

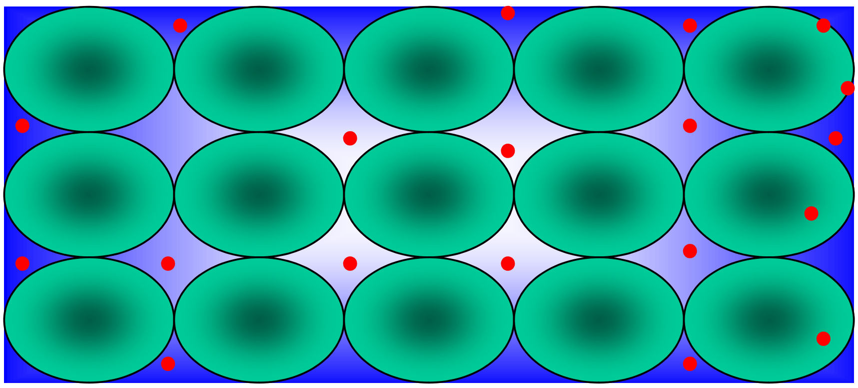


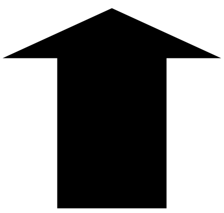
As you heat the metal, the particles vibrate, these vibrations make the adjacent particles vibrate, and so on and so on, the vibrations are passed along the metal and so is the heat. We call this? Conduction

CONDUCTION (For metals)

The outer electrons for metal atoms drift, and are free to move.

When the metal is heated, this 'Sea of electrons' gain kinetic energy and transfer it throughout the metal.



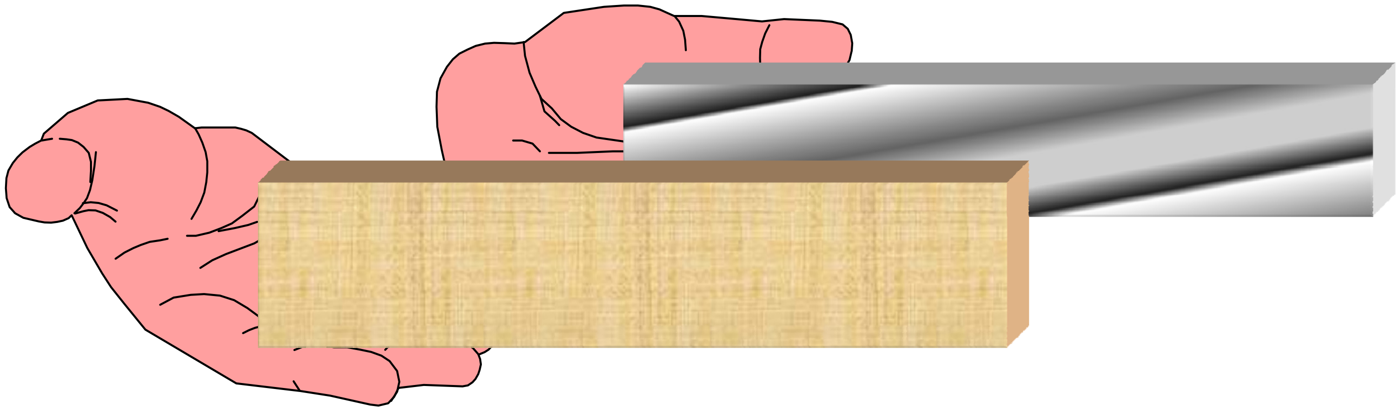


Insulators, such as wood and plastic, do not have this 'Sea of electrons' which is why they do not conduct heat as well as metals.

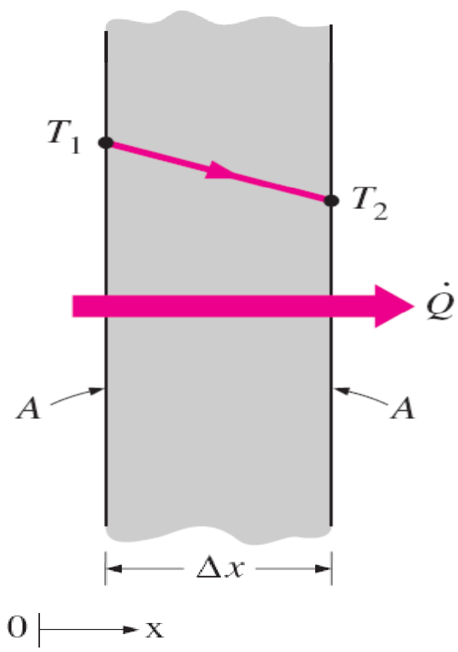
CONDUCTION

Why does metal feel colder than wood, if they are both at the same temperature?

Metal is a conductor, wood is an insulator. The metal conducts the heat away from your hands, the wood does not conduct the heat away from your hands as well as the metal, so the wood feels warmer than the metal.



CONDUCTION



• Consider steady heat conduction through a large plane wall of thickness $\Delta x = L$ and area *A, as shown in Fig.* The temperature difference across the wall is $\Delta T = T_2 - T_1$.

• Experiments have shown that the rate of heat transfer *Q* through the wall is *doubled when the temperature difference across* the wall or the area *A normal to the direction of heat transfer is doubled, but is halved when the wall thickness L is doubled.*

• *Thus the rate of heat conduction through a plane layer is proportional to the temperature difference across the layer and the heat transfer area, but is inversely proportional to the thickness of the layer.*

CONDUCTION

$$\text{Rate of heat conduction} \propto \frac{(\text{Area})(\text{Temperature difference})}{\text{Thickness}}$$

$$\dot{Q}_{\text{cond}} = kA\frac{T_1 - T_2}{\Delta x} = -kA\frac{\Delta T}{\Delta x}$$

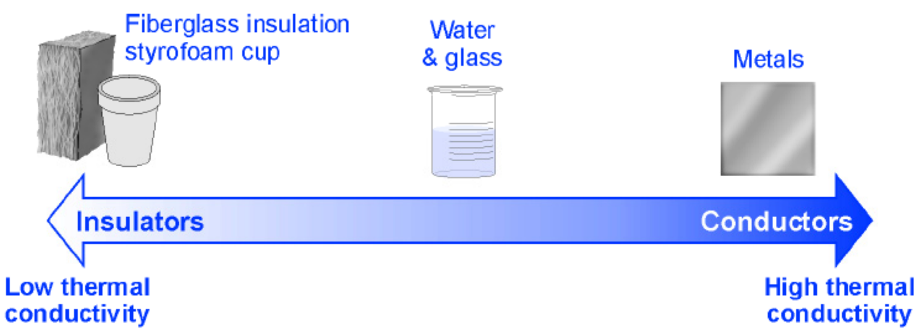
where the constant of proportionality *k is the **thermal conductivity of the*** material, which is a *measure of the ability of a material to conduct heat* (Fig.). In the limiting case of *x →0, the equation above reduces to the* differential form which is called **Fourier's law of heat conduction after J. Fourier, who expressed** it first in his heat transfer text in 1822.

$$\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx}$$

The relation above indicates that the rate of heat conduction in a direction is proportional to the temperature gradient in that direction. Heat is conducted in the direction of decreasing temperature, and the temperature gradient becomes negative when temperature decreases with increasing *x. The negative sign in Eq. ensures that heat* transfer in the positive *x direction is a positive quantity and* the heat transfer area *A is always normal to the direction of heat transfer.*

THERMAL CONDUCTIVITY

- **The thermal conductivity *k is a measure of a material's ability to conduct heat.*** For example, *k 0.608 W/m · °C for water and k 80.2 W/m · °C for iron at room* temperature, which indicates that iron conducts heat more than 100 times faster than water can. Thus we say that water is a poor heat conductor relative to iron, although water is an excellent medium to store thermal energy.
- The **thermal conductivity of a material can be defined as** ***the rate of*** *heat transfer through a unit thickness of the material per unit area per unit temperature difference.* The thermal conductivity of a material is a measure of the ability of the material to conduct heat. A high value for thermal conductivity indicates that the material is a good heat conductor, and a low value indicates that the material is a poor heat conductor or *insulator.*



THERMAL CONDUCTIVITY

- The thermal conductivity of pure copper at room temperature is *k 401 W/m · °C, which indicates that a 1-m-thick copper wall will conduct* heat at a rate of 401 W per m^2 area per °C temperature difference across the wall. The materials such as copper and silver that are good electric conductors are also good heat conductors, and have high values of thermal conductivity. Materials such as rubber, wood are poor conductors of heat and have low conductivity values.
- The thermal conductivities of materials vary over a wide range, the thermal conductivities of gases such as air vary by a factor of 10^4 from those of pure metals such as copper. The pure crystals and metals have the highest thermal conductivities, and gases and insulating materials the lowest.

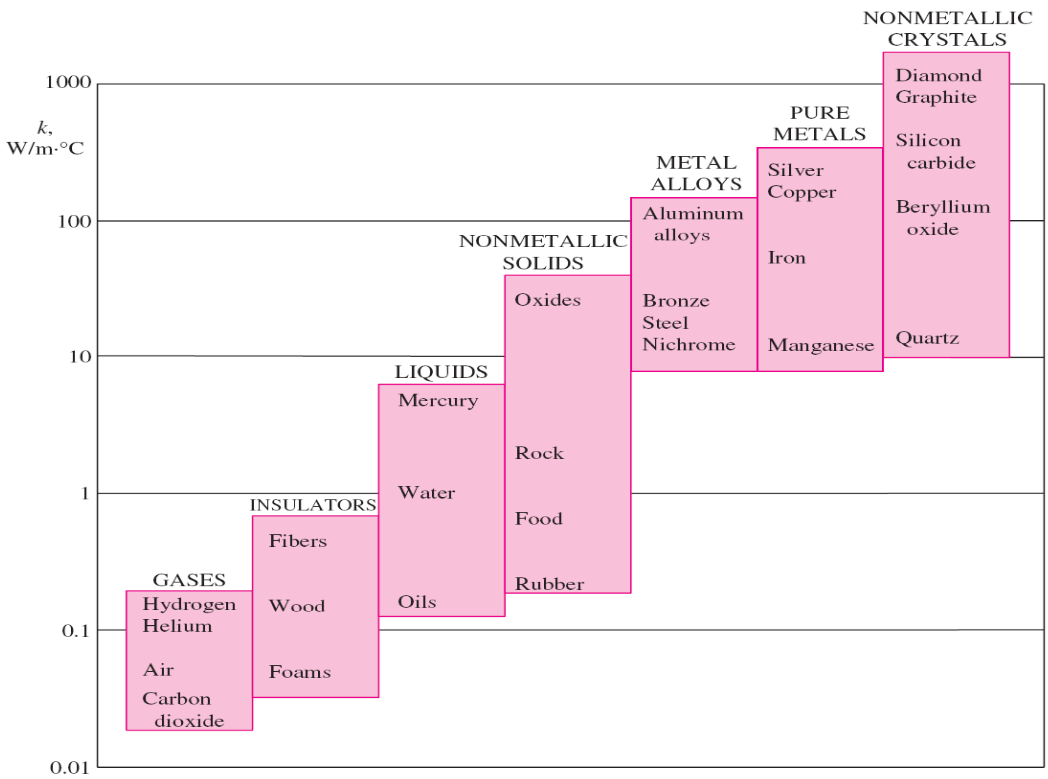
THERMAL CONDUCTIVITY

- Temperature is a measure of the kinetic energies of the particles such as the molecules or atoms of a substance. In a liquid or gas, the kinetic energy of the molecules is due to their random translational motion as well as their vibrational and rotational motions. When two molecules possessing different kinetic energies collide, part of the kinetic energy of the more energetic (higher-temperature) molecule is transferred to the less energetic (lower temperature) molecule, much the same as when two elastic balls of the same mass at different velocities collide, part of the kinetic energy of the faster ball is transferred to the slower one. The higher the temperature, the faster the molecules move and the higher the number of such collisions, and the better the heat transfer.
- The *kinetic theory of gases predicts and the experiments confirm that the* thermal conductivity of gases is proportional to the *square root of the absolute temperature T, and inversely proportional to the square root of the molar mass M.*

THERMAL CONDUCTIVITY

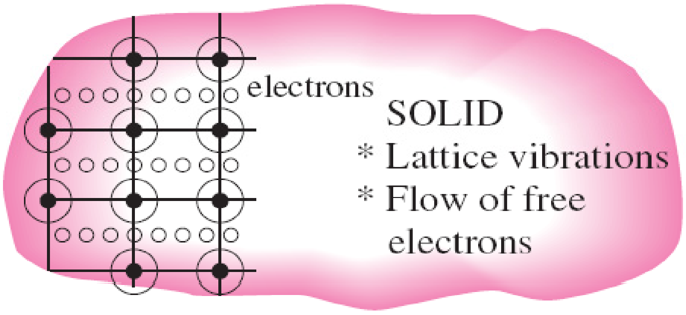
- Therefore, the thermal conductivity of a gas increases with increasing temperature and decreasing molar mass. So it is not surprising that the thermal conductivity of helium (M = *4) is much higher than those of air* (M = *29) and argon* (M *=40).*
- The thermal conductivity of gases is *independent of pressure in a wide range of* pressures encountered in practice.
- The mechanism of heat conduction in a *liquid is complicated by the fact that* the molecules are more closely spaced, and they exert a stronger intermolecular force field. The thermal conductivities of liquids usually lie between those of solids and gases.
- The thermal conductivity of a substance is normally highest in the solid phase and lowest in the gas phase. The thermal conductivities of most liquids decrease with increasing temperature, with water being a notable exception. Like gases, the conductivity of liquids decreases with increasing molar mass.

The range of thermal conductivity of various materials at room temperature.



THERMAL CONDUCTIVITY

- In *solids, heat conduction is due to two effects: the lattice vibrational waves* induced by the vibrational motions of the molecules positioned at relatively fixed positions in a periodic manner called a lattice, and the energy transported via the *free flow of electrons in the solid.*



- The thermal conductivity of a solid is obtained by adding the lattice and electronic components. The relatively high thermal conductivities of pure metals are primarily due to the electronic component. The lattice component of thermal conductivity strongly depends on the way the molecules are arranged. For example, diamond, which is a highly ordered crystalline solid, has the highest known thermal conductivity at room temperature.

THERMAL CONDUCTIVITY

- The thermal conductivities of materials vary with temperature. The variation of thermal conductivity over certain temperature ranges is negligible for some materials, but significant for others. The thermal conductivities of certain solids exhibit dramatic increases at temperatures near absolute zero, when these solids become *superconductors*.

Thermal conductivities of materials vary with temperature

T, K	Copper	Aluminum
100	482	302
200	413	237
300	401	237
400	393	240
600	379	231
800	366	218

THERMAL DIFFUSIVITY

- Another material property that appears in the transient heat conduction analysis is the **thermal diffusivity, which represents how fast heat diffuses** through a material and is defined as

$$\alpha = \frac{\text{Heat conducted}}{\text{Heat stored}} = \frac{k}{\rho C_p} \qquad (m^2/s)$$

- Therefore, the thermal diffusivity of a material can be viewed as the ratio of the heat conducted through the material to the heat stored per unit volume.
- A material that has a high thermal conductivity or a low heat capacity will obviously have a large thermal diffusivity. The larger the thermal diffusivity, the faster the propagation of heat into the medium. A small value of thermal diffusivity means that heat is mostly absorbed by the material and a small amount of heat will be conducted further.

The thermal diffusivities of some materials at room temperature

Material	α, m^2/s*
Silver	149×10^{-6}
Gold	127×10^{-6}
Copper	113×10^{-6}
Aluminum	97.5×10^{-6}
Iron	22.8×10^{-6}
Mercury (l)	4.7×10^{-6}
Marble	1.2×10^{-6}
Ice	1.2×10^{-6}
Concrete	0.75×10^{-6}
Brick	0.52×10^{-6}
Heavy soil (dry)	0.52×10^{-6}
Glass	0.34×10^{-6}
Glass wool	0.23×10^{-6}
Water (l)	0.14×10^{-6}
Beef	0.14×10^{-6}
Wood (oak)	0.13×10^{-6}

*Multiply by 10.76 to convert to ft^2/s.

HEAT CONDUCTION EQUATION

- Heat transfer has *direction as well as magnitude. The rate of heat conduction* in a specified direction is proportional to the *temperature gradient,* which is the change in temperature per unit length in that direction. Heat conduction in a medium, in general, is three-dimensional and time dependent. That is, $T = T(x, y, z, t)$ *and the temperature in a medium* varies with position as well as time.
- Heat conduction in a medium is said to be *steady when the temperature does not vary with time, and unsteady or transient* when it does.
- Heat conduction in a medium is said to be *one-dimensional* when conduction is significant in one dimension only and negligible in the other two dimensions, *two-dimensional when conduction in the third dimension* is negligible, and *three-dimensional when conduction in all dimensions* is significant.

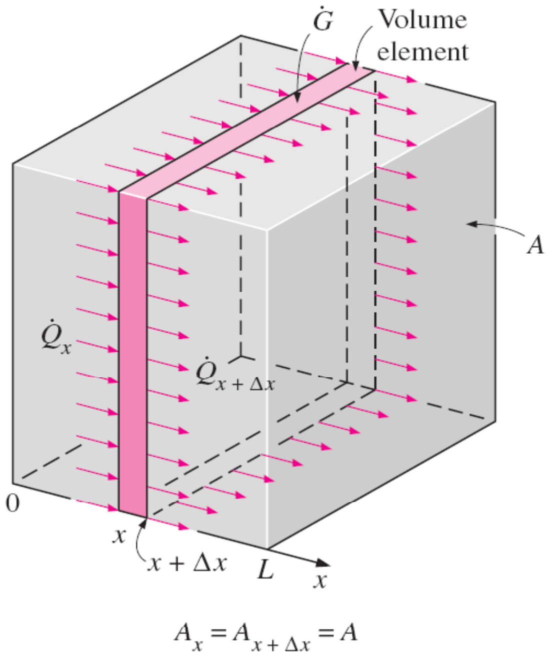
HEAT CONDUCTION EQUATION

- This topic deals with a description of steady, unsteady, and multidimensional heat conduction. Then we derive the differential equation that governs heat conduction in a large plane wall, a long cylinder, and a sphere, and generalize the results to three-dimensional cases in rectangular, cylindrical, and spherical coordinates.
- Heat conduction was defined as the transfer of thermal energy from the more energetic particles of a medium to the adjacent less energetic ones. It was stated that conduction can take place in liquids and gases as well as solids provided that there is no bulk motion involved.
- The driving force for any form of heat transfer is the *temperature difference,* and the larger the temperature difference, the larger the rate of heat transfer. Some heat transfer problems in engineering require the determination of the *temperature distribution (the variation of temperature) throughout the* medium in order to calculate some quantities of interest such as the local heat transfer rate, thermal expansion, and thermal stress at some critical locations at specified times.

(A) ONE-DIMENSIONAL HEAT CONDUCTION EQUATION

Consider heat conduction through a large plane wall such as the wall of a house, the glass of a single pane window, the metal plate at the bottom of a pressing iron, a cast iron steam pipe, a cylindrical nuclear fuel element, an electrical resistance wire, the wall of a spherical container, or a spherical metal ball that is being quenched or tempered.

Heat conduction in these and many other geometries can be approximated as being *one-dimensional* since heat conduction through these geometries will be dominant in one direction and negligible in other directions. Below we will develop the one dimensional heat conduction equation in rectangular, cylindrical, and spherical coordinates.



One-dimensional heat conduction through a volume element in a large plane wall.

(A) HEAT CONDUCTION EQUATION IN A LARGE PLANE WALL

- Consider a thin element of thickness Δx *in a large plane wall, as shown in Figure* . Assume the density of the wall is ρ , the specific heat is *C, and the* area of the wall normal to the direction of heat transfer is *A. An energy balance* on this thin element during a small time interval Δt *can be expressed as*

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } x+\Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

OR

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \text{(A.1)}$$

(A) HEAT CONDUCTION EQUATION IN A LARGE PLANE WALL

- But the change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta x (T_{t+\Delta t} - T_t)$$

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta x \qquad \text{(A.2)}$$

Substituting eq. (A.2) in eq. (A.1)

$$\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{g} A \Delta x = \rho C A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(A.3)}$$

or

$$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(A.4)}$$

(A)HEAT CONDUCTION EQUATION IN A LARGE PLANE WALL

or $$-\frac{1}{A}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(A.5)}$$

Taking the limit as $x \rightarrow 0$ *and* $t \rightarrow 0$ *yields*

$$\frac{1}{A}\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \qquad \text{(A.6)}$$

from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x \rightarrow 0} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{\partial \dot{Q}}{\partial x} = \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right) \qquad \text{(A.7)}$$

the area A *is constant for a plane wall, the one-dimensional transient* heat conduction equation in a plane wall becomes

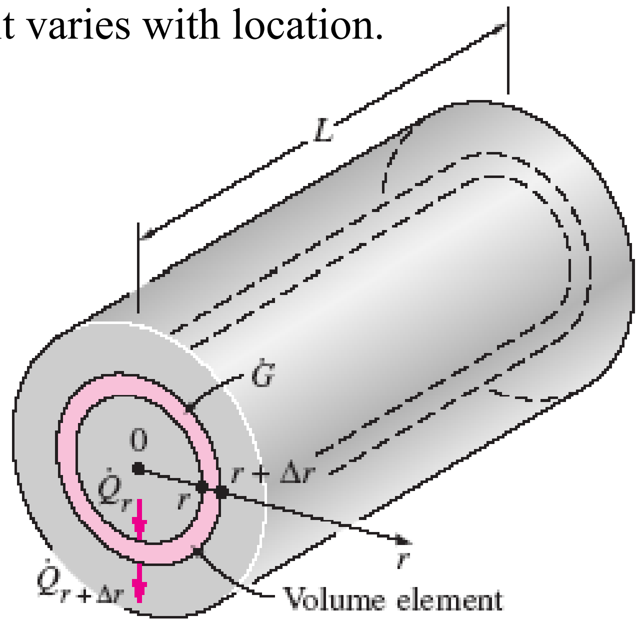
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \qquad \text{(A.8)}$$

(A) HEAT CONDUCTION EQUATION IN A LARGE PLANE WALL

Conditions	Heat transfer equation
Variable thermal conductivity	$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{g}=\rho C\frac{\partial T}{\partial t}$
Constant thermal conductivity	$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$
Steady state	$\frac{d^2T}{dx^2}+\frac{\dot{g}}{k}=0$
Transient state with no heat generation	$\frac{\partial^2 T}{\partial x^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$
Steady state with no heat generation	$\frac{d^2T}{dx^2}=0$

(B) HEAT CONDUCTION EQUATION IN A LONG CYLINDER

Let us consider a thin cylindrical shell element of thickness Δr *in a long cylinder,* as shown in Figure. Assume the density of the cylinder is , the specific heat is *C, and the length is L. The area of the cylinder normal to the* direction of heat transfer at any location is $A = 2\pi rL$ *where r is the value of* the radius at that location. Note that the heat transfer area *A depends on r in* this case, and thus it varies with location.



One-dimensional heat conduction through a volume element in a long cylinder.

- An *energy balance on this thin* cylindrical shell element during a small time interval *t can be expressed as*

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at } r+\Delta r\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of the}\\ \text{element}\end{pmatrix}$$

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \qquad \text{(B.1)}$$

The change in the energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta r (T_{t+\Delta t} - T_t)$$
$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta r \qquad \text{(B.2)}$$

Substituting into Eq. (B.2) in (B.1) we get

$$\dot{Q}_r - \dot{Q}_{r+\Delta r} + \dot{g} A \Delta r = \rho C A \Delta r \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(B.3)}$$

where A =2πrL. You may be tempted to express the area at the middle of the element using the average radius as A= 2π(r +Δ r/2)L. But there is nothing we can gain from this complication since later in the analysis we will take the limit as Δr → 0 and thus the term Δr/2 will drop out. Now dividing the equation above by AΔr give*s*

$$-\frac{1}{A}\frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} \qquad \text{(B.4)}$$

Taking the limit as *r → 0 and t → 0 yields*

$$\frac{1}{A}\frac{\partial}{\partial r}\left(kA\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \qquad \text{(B.5)}$$

from the definition of the derivative and Fourier's law of heat conduction,

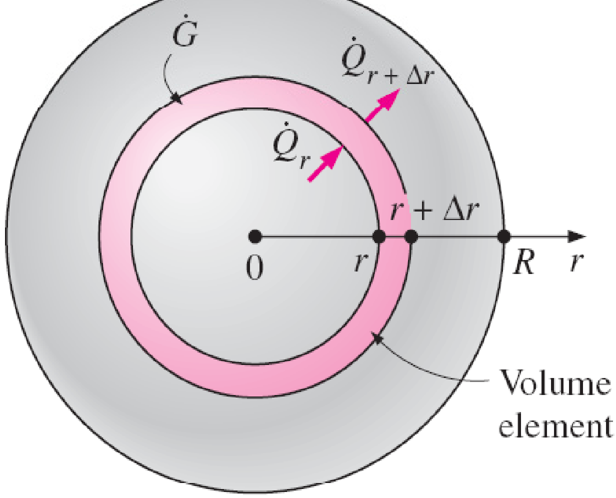
$$\lim_{\Delta r \to 0} \frac{\dot{Q}_{r+\Delta r} - \dot{Q}_r}{\Delta r} = \frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-kA\frac{\partial T}{\partial r}\right) \qquad \text{(B.6)}$$

(B) HEAT CONDUCTION EQUATION IN A LONG CYLINDER

conditions	Heat transfer equation
Variable thermal conductivity	$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$
Constant thermal conductivity	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
Steady state	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$
Transient state with no heat generation	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
Steady state with no heat generation	$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$

(C) HEAT CONDUCTION EQUATION IN A SPHERE

Now consider a sphere with densityρ, specific heat *C, and outer radius R. The* area of the sphere normal to the direction of heat transfer at any location is $A=4\pi r^2$*, where r is the value of the radius at that location. Note that the heat* transfer area *A depends on r in this case also, and thus it varies with location.* By considering a thin spherical shell element of thickness Δr *and repeating* the approach described above for the cylinder by using $A= 4\pi r^2$ *instead of* $A=2\pi rL$*, the one-dimensional transient heat conduction equation for a* sphere is determined to be (Fig.)



One-dimensional heat conduction through a volume element in a sphere.

(C) HEAT CONDUCTION EQUATION IN A SPHERE

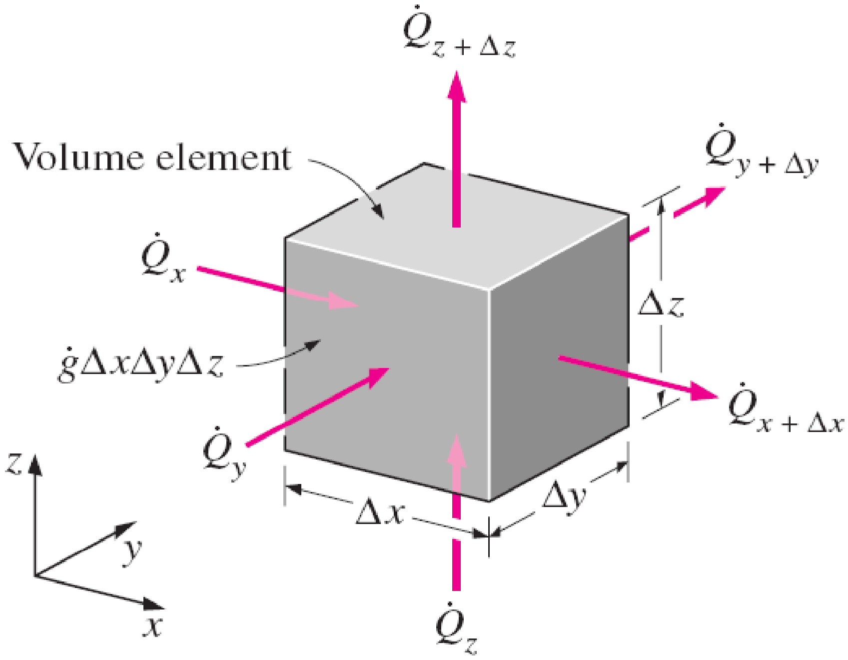
Conditions	Heat transfer equation
Variable thermal conductivity	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k\frac{\partial T}{\partial r}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$
Constant thermal conductivity	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
Steady state	$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + \frac{\dot{g}}{k} = 0$
Transient state with no heat generation	$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$
Steady state with no heat generation	$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$

(D) HEAT CONDUCTION EQUATION IN RECTANGULAR COORDINATE

- In the last section we considered one-dimensional heat conduction and assumed heat conduction in other directions to be negligible. Most heat transfer problems encountered in practice can be approximated as being one dimensional, and we will mostly deal with such problems in this text. However, this is not always the case, and sometimes we need to consider heat transfer in other directions as well. In such cases heat conduction is said to be *multidimensional, and in this section we will develop the governing differential* equation in such systems in rectangular, cylindrical, and spherical coordinate systems.

(D) HEAT CONDUCTION EQUATION IN RECTANGULAR COORDINATE

Consider a small rectangular element of length Δx, *width* Δy, *and height* Δz, as shown in Figure. Assume the density of the body is ρ and the specific heat is *C. An energy balance on this element during a small time interval* Δt can be expressed as



Three-dimensional heat conduction through a rectangular volume element.

(D) HEAT CONDUCTION EQUATION IN RECTANGULAR COORDINATE

$$\begin{pmatrix}\text{Rate of heat}\\ \text{conduction at}\\ x,\ y,\ \text{and}\ z\end{pmatrix} - \begin{pmatrix}\text{Rate of heat}\\ \text{conduction}\\ \text{at}\ x+\Delta x,\\ y+\Delta y,\ \text{and}\ z+\Delta z\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

or

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \quad \text{(D.1)}$$

Noting that the volume of the element is $V_{element}=\Delta x\Delta y\Delta z$, *the change in the* energy content of the element and the rate of heat generation within the element can be expressed as

$$\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C \Delta x \Delta y \Delta z (T_{t+\Delta t} - T_t)$$
$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} \Delta x \Delta y \Delta z \quad \text{(D.2)}$$

Substituting into Eq. (D.2) in (D.1), we get

$$\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{g}\Delta x\Delta y\Delta z = \rho C\Delta x\Delta y\Delta z\frac{T_{t+\Delta t} - T_t}{\Delta t} \quad \text{(D.3)}$$

Dividing by $\Delta x\Delta y\Delta z$ *gives*

$$-\frac{1}{\Delta y\Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} - \frac{1}{\Delta x\Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} - \frac{1}{\Delta x\Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{g} = \rho C\frac{T_{t+\Delta t} - T_t}{\Delta t} \quad \text{(D.4)}$$

Noting that the heat transfer areas of the element for heat conduction in the *x, y, and z directions are* $A_x=\Delta y\Delta z$, $A_y=\Delta x\Delta z$, *and* $A_z=\Delta x\Delta y$, *respectively,* and taking the limit as Δx, Δy, Δz *and* $\Delta t \rightarrow 0$ *yields*

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \quad \text{(D.5)}$$

Since, from the definition of the derivative and Fourier's law of heat conduction,

$$\lim_{\Delta x\rightarrow 0}\frac{1}{\Delta y\Delta z}\frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} = \frac{1}{\Delta y\Delta z}\frac{\partial Q_x}{\partial x} = \frac{1}{\Delta y\Delta z}\frac{\partial}{\partial x}\left(-k\Delta y\Delta z\frac{\partial T}{\partial x}\right) = -\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$$

$$\lim_{\Delta y\rightarrow 0}\frac{1}{\Delta x\Delta z}\frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} = \frac{1}{\Delta x\Delta z}\frac{\partial Q_y}{\partial y} = \frac{1}{\Delta x\Delta z}\frac{\partial}{\partial y}\left(-k\Delta x\Delta z\frac{\partial T}{\partial y}\right) = -\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \quad \text{(D.6)}$$

$$\lim_{\Delta z\rightarrow 0}\frac{1}{\Delta x\Delta y}\frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} = \frac{1}{\Delta x\Delta y}\frac{\partial Q_z}{\partial z} = \frac{1}{\Delta x\Delta y}\frac{\partial}{\partial z}\left(-k\Delta x\Delta y\frac{\partial T}{\partial z}\right) = -\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

(D) HEAT CONDUCTION EQUATION IN RECTANGULAR COORDINATE

The general heat conduction equation in rectangular coordinates. In the case of constant thermal conductivity, it reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(D.7)}$$

(1) *Steady-state:* (called the **Poisson equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0 \qquad \text{(D.8)}$$

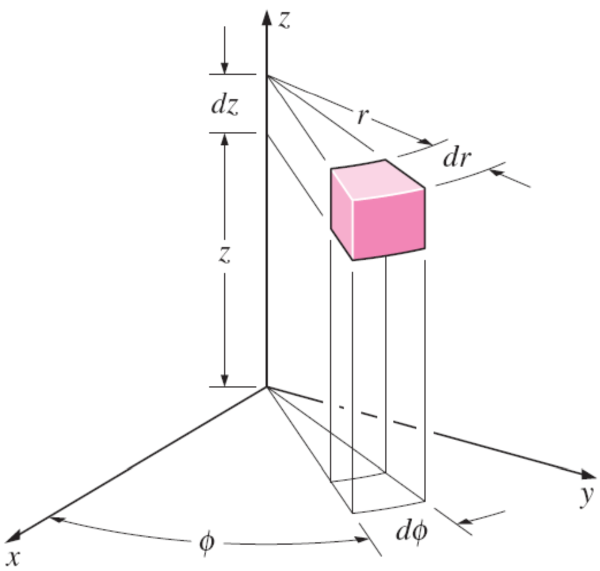
(2) *Transient, no heat generation:* (called the **diffusion equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{(D.9)}$$

(3) *Steady-state, no heat generation:* (called the **Laplace equation**)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \text{(D.10)}$$

(D) HEAT CONDUCTION EQUATION IN CYLINDRICAL COORDINATE



A differential volume element in cylindrical coordinates.

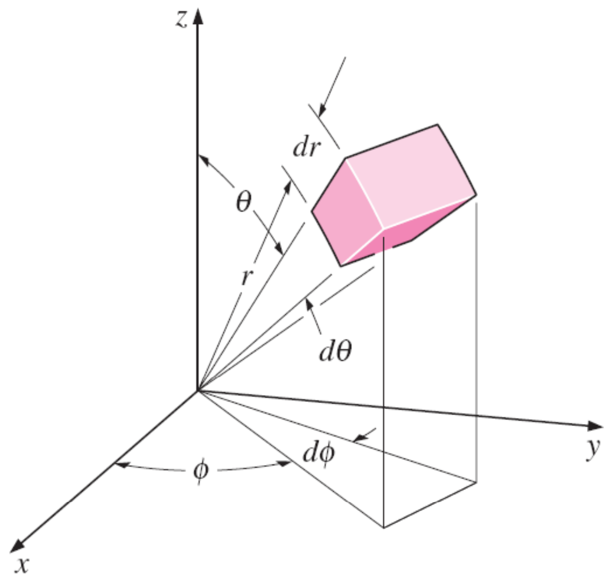
The general heat conduction equation in cylindrical coordinates can be obtained from an energy balance on a volume element in cylindrical coordinates, shown in Figure, by following the steps just outlined. It can also be obtained directly from Eq. (D.5) by coordinate transformation using the following relations between the coordinates of a point in rectangular and cylindrical coordinate systems:

$$x = r\cos\phi, \quad y = r\sin\phi, \quad \text{and} \quad z = z$$

And the final equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(kr\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t} \qquad \text{(D.11)}$$

(D) HEAT CONDUCTION EQUATION IN SPHERICAL COORDINATE



A differential volume element in spherical coordinates.

The general heat conduction equations in spherical coordinates can be obtained from an energy balance on a volume element in spherical coordinates, shown in Figure, by following the steps outlined above. It can also be obtained directly from Eq. (D.5) by coordinate transformation using the following relations between the coordinates of a point in rectangular and spherical coordinate systems:

r= cosϕ sinθ , y= r sinϕ sinθ , and z = cosθ

Again after lengthy manipulations,

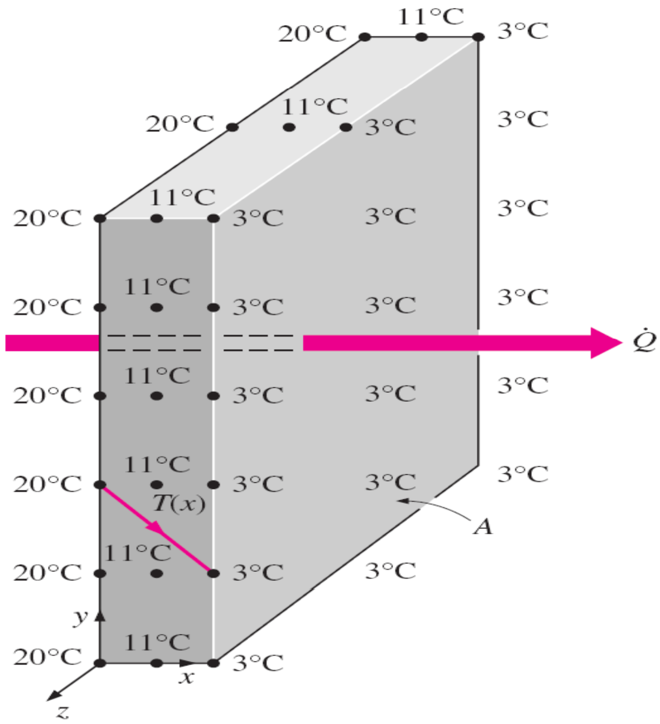
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{g}=\rho C\frac{\partial T}{\partial t} \quad \text{(D.12)}$$

(D) HEAT CONDUCTION EQUATION

- Obtaining analytical solutions to these differential equations requires a knowledge of the solution techniques of partial differential equations, which is beyond the scope of this introductory text. Here we limit our consideration to one-dimensional steady-state cases or lumped systems, since they result in ordinary differential equations.

(E) STEADY-STATE CONDUCTION IN ONE DIMENSION FOR PLANE WALL

Consider steady heat conduction through the walls of a house during a winter day. We know that heat is continuously lost to the outdoors through the wall. We intuitively feel that heat transfer through the wall is in the *normal direction* to the wall surface, and no significant heat transfer takes place in the wall in other directions (Fig.).



Heat flow through a wall is one dimensional when the temperature of the wall varies in one direction only.

(E) STEADY-STATE CONDUCTION IN ONE DIMENSION FOR PLANE WALL

- heat transfer in a certain direction is driven by the *temperature gradient in that direction. There will be no heat transfer in a direction in which* there is no change in temperature. Temperature measurements at several locations on the inner or outer wall surface will confirm that a wall surface is nearly *isothermal. That is, the temperatures at the top and bottom of a wall* surface as well as at the right or left ends are almost the same. Therefore, there will be no heat transfer through the wall from the top to the bottom, or from left to right, but there will be considerable temperature difference between the inner and the outer surfaces of the wall, and thus significant heat transfer in the direction from the inner surface to the outer one.
- The small thickness of the wall causes the temperature gradient in that direction to be large. Further, if the air temperatures in and outside the house remain constant, then heat transfer through the wall of a house can be modeled as *steady and one-dimensional. The temperature of the wall in this case* will depend on one direction only (say the *x-direction) and can be expressed* as *T(x)*.

(E) STEADY-STATE CONDUCTION IN ONE DIMENSION FOR PLANE WALL

- let us consider that heat transfer is the only energy interaction involved in this case and there is no heat generation, the *energy balance for the wall can be expressed* as

$$\begin{pmatrix}\text{Rate of} \\ \text{heat transfer} \\ \text{into the wall}\end{pmatrix} - \begin{pmatrix}\text{Rate of} \\ \text{heat transfer} \\ \text{out of the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of change} \\ \text{of the energy} \\ \text{of the wall}\end{pmatrix}$$

or

$$\dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} = \frac{dE_{\text{wall}}}{dt} \qquad \text{(E.1)}$$

And for steady-state operation

$$dE_{\text{wall}}/dt = 0 \qquad \text{(E.2)}$$

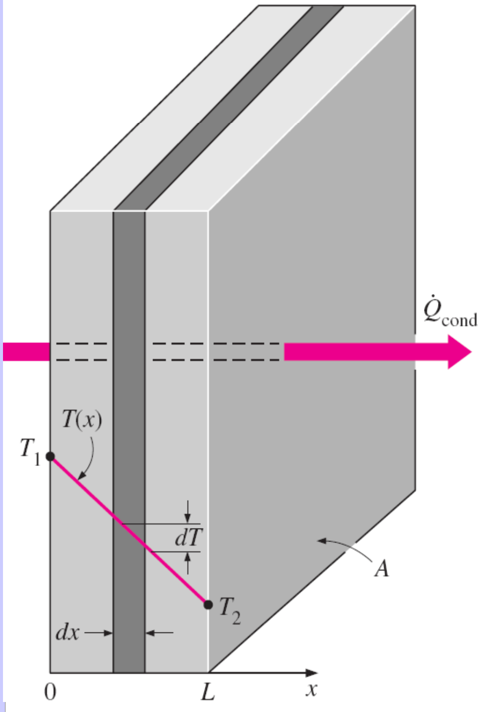
(E) STEADY-STATE CONDUCTION IN ONE DIMENSION FOR PLANE WALL

- Let us consider a plane wall of thickness *L and average thermal conductivity k.* The two surfaces of the wall are maintained at constant temperatures of T_1 *and* T_2*. For one-dimensional steady heat conduction through the wall,* we have *T(x). Then Fourier's law of heat conduction for the wall can be* expressed as

$$\dot{Q}_{\text{cond, wall}} = -kA\frac{dT}{dx} \qquad \text{(E.3)}$$

where the rate of conduction heat transfer $Q_{\text{cond wall}}$ and the wall area *A are* constant. Thus we have *dT/dx constant, which means that the temperature through the wall varies linearly with x. That is, the temperature distribution in* the wall under steady conditions is a *straight line as shown in next figure.*

(E) STEADY-STATE CONDUCTION IN ONE DIMENSION FOR PLANE WALL



Under steady conditions, the temperature distribution in a plane wall is a straight line.

Separating the variables in the above equation and integrating from *x= 0*, where *T(0) = T_1, to x= L, where T(L) = T_2, we get*

$$\int_{x=0}^{L} \dot{Q}_{\text{cond, wall}}\, dx = -\int_{T=T_1}^{T_2} kA\, dT \qquad \text{(E.4)}$$

Performing the integrations and rearranging gives

$$\dot{Q}_{\text{cond, wall}} = kA\frac{T_1 - T_2}{L} \qquad \text{(E.5)}$$

the rate of heat conduction through a plane wall is proportional to the average thermal conductivity, the wall area, and the temperature difference, but is inversely proportional to the wall thickness. Also, once the rate of heat conduction is available, the temperature T(x) at any location x can be determined by replacing T_2 in Eq. (E.5)* by *T, and L by x.

(E) THERMAL RESISTANCE

- Equation (E.5) for heat conduction through a plane wall can be rearranged as

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \qquad \text{(E.6)}$$

Where R_{wall} is the thermal resistance of the wall against heat conduction.

$$R_{\text{wall}} = \frac{L}{kA} \qquad \text{(E.7)}$$

the thermal resistance of a medium depends on the *geometry and the thermal properties of the medium.* The equation above for heat flow is analogous to the relation for *electric current flow I, expressed as*

$$I = \frac{\mathbf{V}_1 - \mathbf{V}_2}{R_e} \qquad \text{(E.8)}$$

(E) THERMAL RESISTANCE

- where $R_e = L/\sigma_e A$ *is the electric resistance and* ***($V_1 - V_2$) is the voltage difference*** across the resistance (σ_e *is the electrical conductivity). Thus, the rate of heat transfer through a layer corresponds to the electric current, the thermal resistance corresponds to electrical resistance, and the temperature difference* corresponds to *voltage difference across the layer*
- Consider convection heat transfer from a solid surface of area A_s *and temperature* T_s *to a fluid whose temperature sufficiently far from the surface is* T_∞, with a convection heat transfer coefficient h. *Newton's law of cooling for convection* heat transfer rate *can be rearranged as*

$$\dot{Q}_{\text{conv}} = \frac{T_s - T_\infty}{R_{\text{conv}}} \qquad \text{(E.9)}$$

Where convection resistance is

$$R_{\text{conv}} = \frac{1}{hA_s} \qquad \text{(E.10)}$$

(E) THERMAL RESISTANCE

- When the wall is surrounded by a gas, the *radiation effects, which we have* ignored so far, can be significant and may need to be considered. The rate of radiation heat transfer between a surface of emissivity and area A_s *at temperature* T_s *and the surrounding surfaces at some average temperature* T_{surr} *can* be expressed as

$$\dot{Q}_{\text{rad}} = \varepsilon\sigma A_s(T_s^4 - T_{\text{surr}}^4) = h_{\text{rad}} A_s(T_s - T_{\text{surr}}) = \frac{T_s - T_{\text{surr}}}{R_{\text{rad}}} \qquad \text{(E.11)}$$

Where thermal resistance again radiation is

$$R_{\text{rad}} = \frac{1}{h_{\text{rad}} A_s} \qquad \text{(E.12)}$$

And **radiation heat transfer coefficient**

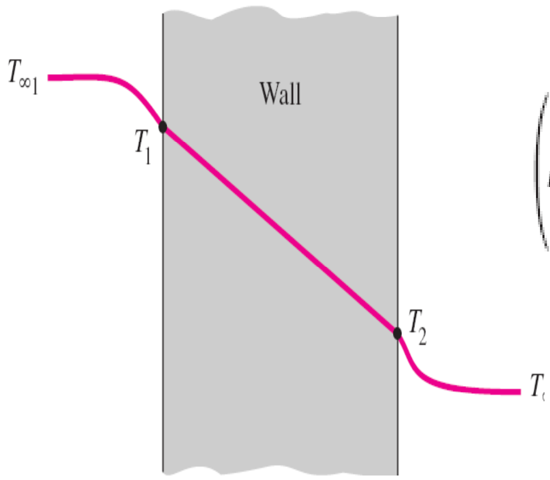
$$h_{\text{rad}} = \frac{\dot{Q}_{\text{rad}}}{A_s(T_s - T_{\text{surr}})} = \varepsilon\sigma(T_s^2 + T_{\text{surr}}^2)(T_s + T_{\text{surr}}) \qquad \text{(E.13)}$$

- A surface exposed to the surrounding air involves convection and radiation simultaneously, and the total heat transfer at the surface is determined by adding (or subtracting, if in the opposite direction) the radiation and convection components.
- where $h_{combined}$ *is the* ***combined heat transfer coefficient***

$$h_{\text{combined}} = h_{\text{conv}} + h_{\text{rad}} \quad \text{w/m}^2\text{.K} \qquad \text{(E.14)}$$

Now consider steady one-dimensional heat flow through a plane wall of thickness *L, area A, and thermal conductivity k that is exposed to convection on* both sides to fluids at temperatures $T_{\infty 1}$ *and* $T_{\infty 2}$ *with heat transfer coefficients* h_1 *and* h_2*, respectively, as shown in next Figure.*

where $T_{\infty 2} < T_{\infty 1}$



The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides,

Under steady state condition

$$\begin{pmatrix}\text{Rate of}\\ \text{heat convection}\\ \text{into the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \text{heat conduction}\\ \text{through the wall}\end{pmatrix} = \begin{pmatrix}\text{Rate of}\\ \text{heat convection}\\ \text{from the wall}\end{pmatrix}$$

or

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A}$$

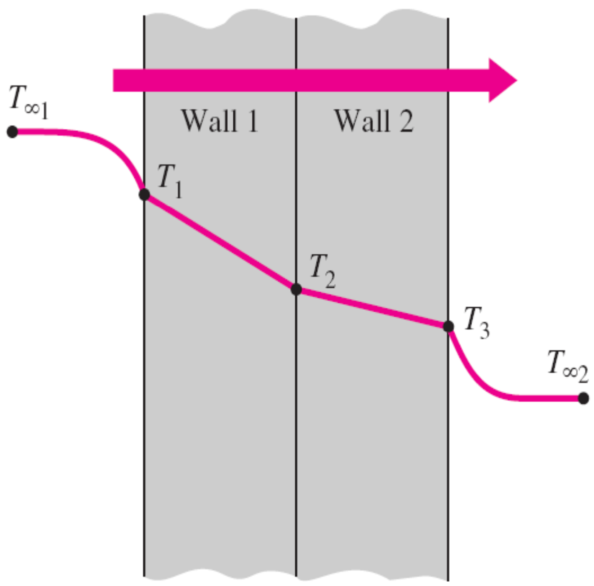
$$= \frac{T_{\infty 1} - T_1}{R_{\text{conv, 1}}} = \frac{T_1 - T_2}{R_{\text{wall}}} = \frac{T_2 - T_{\infty 2}}{R_{\text{conv, 2}}} \qquad \text{(E.15)}$$

There fore total thermal resistance is as follows

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall}} + R_{\text{conv, 2}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \qquad \text{(E.16)}$$

(F) HEAT CONDUCTION THROUGH MULTI-LAYER WALL

Total thermal resistance



$$R_{total} = R_{conv,\,1} + R_{wall,\,1} + R_{wall,\,2} + R_{conv,\,2}$$
$$= \frac{1}{h_1 A} + \frac{L_1}{k_1 A} + \frac{L_2}{k_2 A} + \frac{1}{h_2 A} \qquad \text{(F.1)}$$

To find T_1: $\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{conv,1}}$

To find T_2: $\dot{Q} = \frac{T_{\infty 1} - T_2}{R_{conv,1} + R_1}$ (F.2)

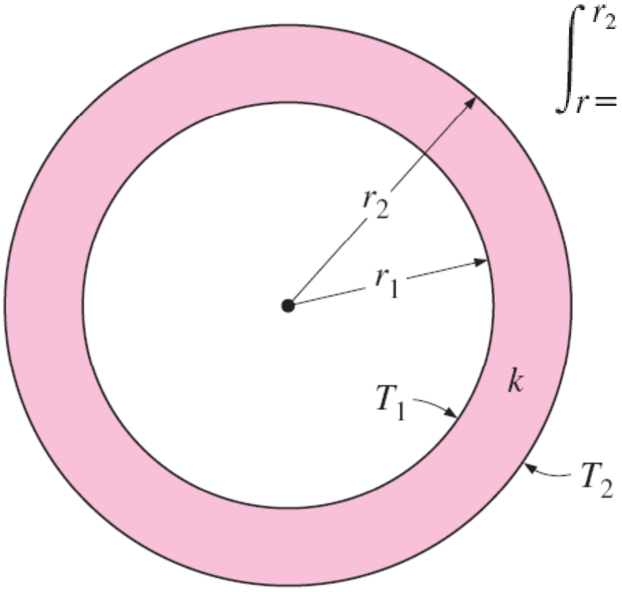
To find T_3: $\dot{Q} = \frac{T_3 - T_{\infty 2}}{R_{conv,2}}$

(G) S-S HEAT CONDUCTION THROUGH CYLINDER

- In *steady operation, there is no change in the temperature of the pipe with* time at any point. Therefore, the rate of heat transfer into the pipe must be equal to the rate of heat transfer out of it. In other words, heat transfer through the pipe must be constant, $Q_{cond,\,cyl}$ =constant.
- Consider a long cylindrical layer (such as a circular pipe) of inner radius r_1, outer radius r_2, *length L, and average thermal conductivity k (Fig.). The* two surfaces of the cylindrical layer are maintained at constant temperatures T_1 *and* T_2. *There is no heat generation in the layer and the thermal conductivity* is constant. For one-dimensional heat conduction through the cylindrical layer, we have *T(r). Then Fourier's law of heat conduction for heat transfer* through the cylindrical layer can be expressed as

$$\dot{Q}_{cond,\,cyl} = -kA\frac{dT}{dr} \qquad \text{(G.1)}$$

(G) S-S HEAT CONDUCTION THROUGH CYLINDER



A long cylindrical pipe

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A}\, dr = -\int_{T=T_1}^{T_2} k\, dT \qquad (G.2)$$

Substituting $A = 2\pi rL$ *and performing the integrations give*

$$\dot{Q}_{\text{cond, cyl}} = 2\pi Lk \frac{T_1 - T_2}{\ln(r_2/r_1)} \qquad (G.3)$$

Thermal conduction resistance

$$R_{\text{cyl}} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius/Inner radius})}{2\pi \times (\text{Length}) \times (\text{Thermal conductivity})} \qquad (G.4)$$

(G) S-S HEAT CONDUCTION THROUGH SPHERE

In case of sphere the area $A=4\pi r^2$. now integrating eq. (G.1) we get

$$\dot{Q}_{\text{cond, sph}} = \frac{T_1 - T_2}{R_{\text{sph}}} \qquad (G.5)$$

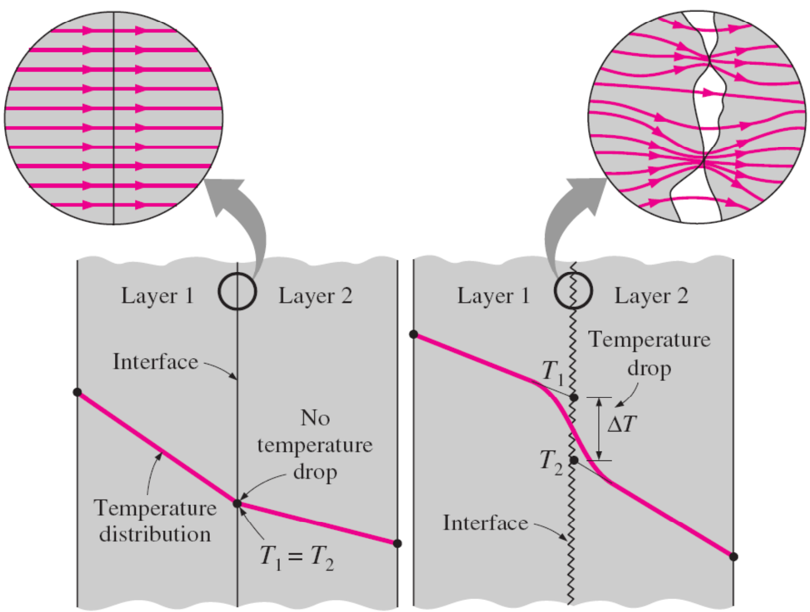
And thermal resistance is

$$R_{\text{sph}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{Outer radius} - \text{Inner radius}}{4\pi(\text{Outer radius})(\text{Inner radius})(\text{Thermal conductivity})} \qquad (G.6)$$

Boundary conditions for solving s-s heat transfer equations

- Dirichlet condition, or a boundary condition of the first kind : fixed surface temperature.
- Neumann condition, or a boundary condition of the second kind: Constant heat flux at surface. For perfectly insulated, or adiabatic system the heat flux at surface is zero.
- Convection Surface condition: amount of heat convected is equal to amount of heat conducted.
- For solving steady-state heat conduction boundary condition if sufficient but for unsteady-state heat conduction both boundary condition and initial conditions are needed.

THERMAL CONTACT RESISTANCE



Ideal (perfect) thermal contact   Actual (imperfect) thermal contact

When two such surfaces are pressed against each other, the peaks will form good material contact but the valleys will form voids filled with air. As a result, an interface will contain numerous *air gaps of varying sizes that act as insulation because of the low thermal conductivity of air. Thus, an interface* offers some resistance to heat transfer, and this resistance per unit interface area is called the **thermal contact resistance.**

Temperature distribution and heat flow lines along two solid plates pressed against each other for the case of perfect and imperfect contact.

(H)THERMAL INSULATION

- Insulations are used to decrease heat flow and to decrease surface temperatures. These materials are found in a variety of forms, typically *loose fill*, *batt*, and *rigid*.
- Even a gas, like air, can be a good insulator if it can be kept from moving when it is heated or cooled. A vacuum is an excellent insulator.
- Usually, though, the engineering approach to insulation is the addition of a low-conducting material to the surface. While there are many chemical forms, costs, and maximum operating temperatures of common forms of insulations, it seems that when a higher operating temperature is required, many times the thermal conductivity and cost of the insulation will also be higher.
- Often, commercial insulation systems designed for high-temperature operation use a layered approach. Temperature tolerance may be critical. Perhaps a refractory is applied in the highest temperature region, an intermediate-temperature foam insulation is used in the middle section, and a high-performance, low temperature insulation is used on the outer side near ambient conditions.

(H)THERMAL INSULATION

Insulating material	Maximum operating temp (^{0}C)	Thermal conductivity (W/m^{2}K)
Loose-fill insulations like 1. milled alumina-silica 2. perlite	 1260 980	 0.1-0.2 0.05 -1.5
Batt-type insulation like 1. Glass-fibre		 0.03 -0.06
Rigid insulations like 1. Polyurethane foam 2. high-alumina brick	 120 1760	 0.02 2

- a

CRITICAL RADIUS OF INSULATION

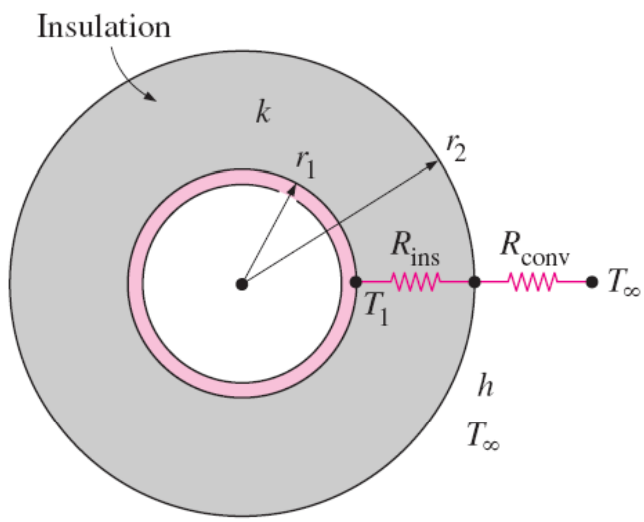
- We know that adding more insulation to a wall or to the attic always decreases heat transfer. The thicker the insulation, the lower the heat transfer rate. This is expected, since the heat transfer area *A is constant, and adding insulation* always increases the thermal resistance of the wall without increasing the convection resistance.
- Adding insulation to a cylindrical pipe or a spherical shell, however, is a different matter. The additional insulation increases the conduction resistance of the insulation layer but decreases the convection resistance of the surface because of the increase in the outer surface area for convection. The heat transfer from the pipe may increase or decrease, depending on which effect dominates.

CRITICAL RADIUS OF INSULATION of a cylindrical pipe

- For the given fig. the rate of heat-loss

From the insulated cylindrical pipe is given by the following equation:

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{ins} + R_{conv}} = \frac{T_1 - T_\infty}{\dfrac{\ln(r_2/r_1)}{2\pi Lk} + \dfrac{1}{h(2\pi r_2 L)}}$$



An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

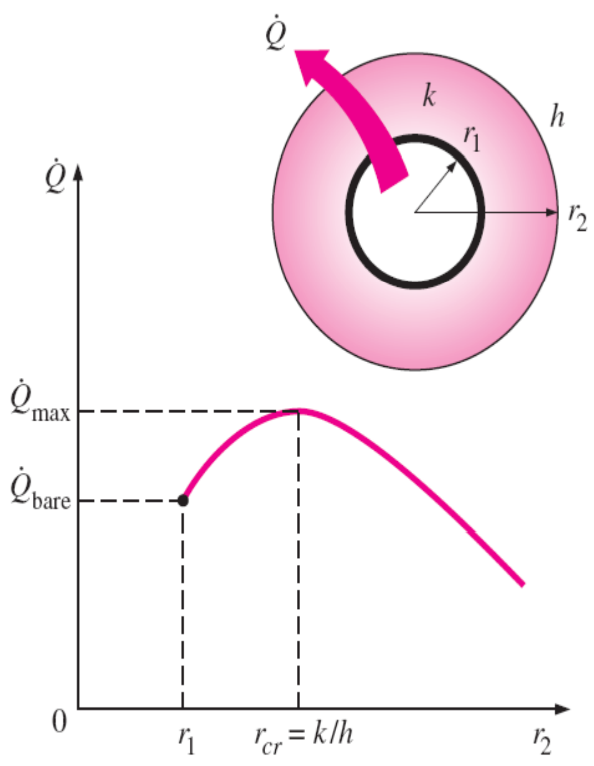
- The variation of $\dot{Q}$

with the outer radius of the insulation r_2 *is plotted in* Fig. below. The value of r_2 *at which* $\dot{Q}$ reaches a maximum is determined from the requirement that $d\dot{Q}/dr_2=0$ *(zero slope). Performing the differentiation* and solving for r_2 *yields the* ***critical radius of insulation for a cylindrical*** body to be

$$r_{cr,\ cylinder} = \frac{k}{h}$$

CRITICAL RADIUS OF INSULATION

the critical radius of insulation depends on the thermal conductivity of the insulation *k and the external convection heat transfer coefficient h.* The rate of heat transfer from the cylinder increases with the addition of insulation for $r_2 < r_{cr}$, *reaches maximum when* $r_2 = r_{cr}$, *and starts to decrease for* $r_2 > r_{cr}$. *Thus, insulating the pipe may actually increase the rate of heat transfer* from the pipe instead of decreasing it when $r_2 < r_{cr}$.



CRITICAL RADIUS OF INSULATION

- **The critical radius of insulation when insulating hot water pipes or even hot water tanks.**
- The value of the critical radius r_{cr} *will be the largest when k is large and h is* small. Noting that the lowest value of *h encountered in practice is about* 5 W/m² · °C for the case of natural convection of gases, and that the thermal conductivity of common insulating materials is about 0.05 W/m² · °C, the largest value of the critical radius we are likely to encounter is

$$r_{cr,\,max} = \frac{k_{max,\,insulation}}{h_{min}} \approx \frac{0.05\ \text{W/m}\cdot°\text{C}}{5\ \text{W/m}^2\cdot°\text{C}} = 0.01\ \text{m} = 1\ \text{cm}$$

- This value would be even smaller when the radiation effects are considered. The critical radius would be much less in forced convection, often less than 1 mm, because of much larger *h values associated with forced convection.* Therefore, we can insulate hot water or steam pipes freely without worrying about the possibility of increasing the heat transfer by insulating the pipes.

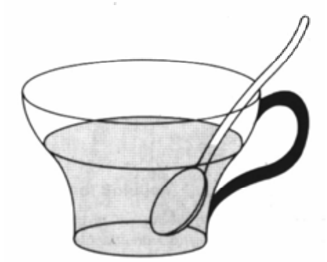
CRITICAL RADIUS OF INSULATION

- The radius of electric wires may be smaller than the critical radius. Therefore, the plastic electrical insulation may actually *enhance the heat transfer* from electric wires and thus keep their steady operating temperatures at lower and thus safer levels.
- The discussions above can be repeated for a sphere, and it can be shown in a similar manner that the critical radius of insulation for a spherical shell is

$$r_{\text{cr, sphere}} = \frac{2k}{h}$$

(I)EXTENDED SURFACE

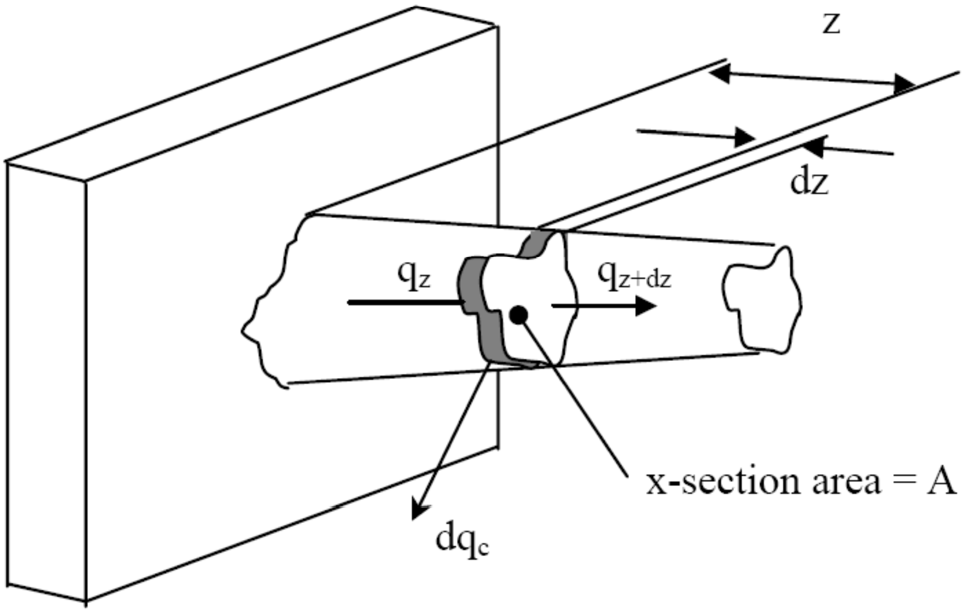
- One example of an extended surface is a spoon placed in a cup of hot coffee. The handle extended beyond the hot coffee. Heat is conducted along the spoon handle, causing the handle to become warmer than the surrounding air. The heat conducted to the handle is then transfer to the air by convection.



A spoon and a cup of coffee.

- The purpose of adding an extended surface is to help dissipate heat.
- Fins are usually added to a heat transfer device to increase the rate of heat removal. This is because of the increase of the heat transfer area.
- With Liquid /Gas HX's very often the heat transfer coefficient on the liquid side is much greater than that on the gas side. Fins would then be used on the gas side so that the resistance to heat transfer was approximately the same on both sides.

General equation for extended surface



An extended surface of arbitrary shape and cross section

General equation for extended surface

- Apply the first law of thermodynamics to the system:

Rate of energy convection out from the control	⇒	Rate of energy conducted into the control volume	−	Rate of energy conducted out off the control volume

$$dq_c = dq_z - dq_{z+dz} \quad (I.1)$$

$$dq_c = dq_z - \left(dq_z + \frac{dq_z}{dz} dz \right) \quad (I.2)$$

$$dq_c = \frac{dq_z}{dz} dz \quad (I.3)$$

- From Fourier law of heat conduction

$$q_x = -k \cdot A \cdot \frac{dT}{dz} \quad (I.4)$$

General equation for extended surface

- Where A is the cross-section and varies along the length, then:

$$\frac{dq_z}{dz} = -k \cdot \frac{d}{dz}\left[A \cdot \frac{dT}{dz}\right] \qquad (I.5)$$

- For convection:

$$dq_c = \bar{h}_c \cdot dA_s (T - T_\infty) \qquad (I.6)$$

- By substituting equations (I.5) and (I.6) into equation (I.3) then:

$$\bar{h}_c \cdot dA_s (T - T_\infty) = -k \cdot \frac{d}{dz}\left[A \cdot \frac{dT}{dz}\right] dz \qquad (I.7)$$

- Then divide by k.dz:

$$\frac{dA}{dz}\frac{dT}{dz} + A\frac{d^2T}{dZ^2} = \frac{\bar{h}_c}{k}\frac{dA_s}{dz}(T - T_\infty) \qquad (I.8)$$

General equation for extended surface

$$\frac{d^2T}{dZ^2} + \frac{1}{A}\frac{dA}{dz}\frac{dT}{dz} - \frac{\bar{h}_c}{k}\frac{1}{A}\frac{dA_s}{dz}(T - T_\infty) = 0 \qquad (I.9)$$

- If we define: $\theta = T - T_\infty$ (I.10)

- Then $\dfrac{dT}{dZ} = \dfrac{d\theta}{dz}$ (I.11)

  and $\dfrac{d^2T}{dZ^2} = \dfrac{d^2\theta}{dz^2}$ (I.12)

- Substitute equations (I.10) to (I.12) into (I.9) then:

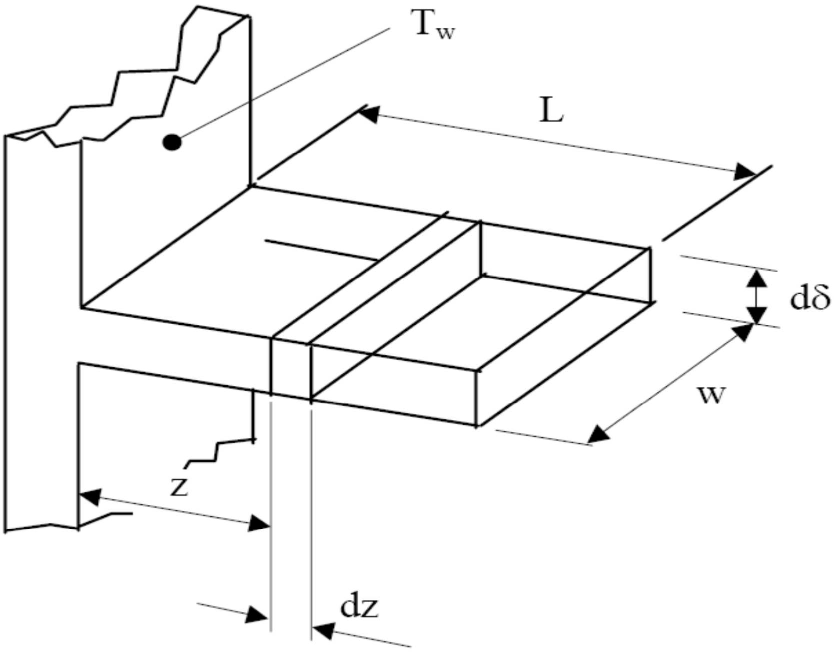
$$\frac{d^2\theta}{dz^2} + \frac{1}{A}\frac{dA}{dz}\frac{d\theta}{dz} - \frac{\bar{h}_c}{k}\frac{1}{A}\frac{dA_s}{dz}\theta = 0 \qquad (I.13)$$

- Equation (I.13) is the general equation for extended surface.

Uniform fin

- Assumptions used in the analysis

1. Fin is uniform or constant cross-sectional area.
2. There is no heat convection out at the fin tip. The fin tip is insulated or the area is very small compared with the total surface area.



A uniform fin

Uniform fin

- Based on these assumptions equation (I.13) is simplified to:

$$\frac{d^2\theta}{dz^2} - \frac{\bar{h}_c}{k}\frac{1}{A}\frac{dA_s}{dz}\theta = 0 \qquad (I.14)$$

- If we define pin perimeter as: $P = \frac{dA_s}{dz}$ (I.15)

- Therefore, equation (I.14) is: $\frac{d^2\theta}{dz^2} - \frac{\bar{h}_c \cdot P}{k \cdot A}\theta = 0$ (I.16)

- Then we define

$$m = \sqrt{\frac{\bar{h}_c \cdot P}{k \cdot A}} \qquad (I.17)$$

- Equation (I.16) is then become

$$\frac{d^2\theta}{dz^2} - m^2 \cdot \theta = 0 \qquad (I.18)$$

Uniform fin

- By solving equation (I.18), we have:

$$\theta = C_1 \cosh(m \cdot Z) + C_2 \sinh(m \cdot z) \qquad (I.19)$$

Or

$$\theta = C_3 e^{mz} + C_4 e^{-mz} \qquad (I.20)$$

- As we assume that the fin's tip is insulated thus:

at root $z = 0, \quad T = T_w \Longrightarrow \theta_w = T_w - T_\infty$

At tip $z = L, \quad \frac{dT}{dz} = 0 \Longrightarrow \frac{d\theta}{dz} = 0$

- from equation (I.19), at root

$$\cosh(0) = 1 \text{ and } \sinh(0) = 0,$$

Thus $C_1 = \theta_w$ (I.21)

Then $\theta = \theta_w \cosh(m \cdot z) + C_2 \sinh(m \cdot z)$ (I.22)

Uniform fin

- By differentiating the above equation

$$\frac{d\theta}{dz} = \theta_w m \cdot \sinh(m \cdot z) + C_2 m \cdot \cosh(m \cdot z) \qquad (I.23)$$

- from equation (I.23), at root

$$0 = \theta_w \sinh(m \cdot L) + C_2 m \cdot \cosh(m \cdot L) \qquad (I.24)$$

- Thus

$$C_2 = -\frac{\theta_w \sinh(m \cdot L)}{\cosh(m \cdot L)} \qquad (I.25)$$

The solution is then

$$\frac{T_z - T_\infty}{T_w - T_\infty} = \frac{\theta}{\theta_w} = \frac{\cosh(m \cdot L)\cosh(m \cdot z) - \sinh(m \cdot L)\sinh(m \cdot z)}{\cosh(m \cdot L)} \qquad (I.26)$$

$$= \frac{\cosh[m \cdot L(1 - z/L)]}{\cosh(m \cdot L)} \qquad (I.27)$$

Uniform fin

- Equation (I.27) can be used to determine temperature (T_z) at distance z from the root. In order to fine the heat rejected out from the fin, Fourier's law of heat conduction may be applied. We also know that, the heat conducted through the root is equal to the heat rejected out by convection, thus:

$$q_z = -k \cdot A \cdot \left.\frac{dT}{dz}\right|_{z=0} \tag{I.28}$$

$$= -k \cdot A \cdot \left.\frac{d\theta}{dz}\right|_{z=0} \tag{I.29}$$

$$= -\frac{k \cdot A \cdot \theta_w}{\cosh(m \cdot L)} \times \left[m \cdot \cosh(m \cdot L)\sinh(m \cdot z) - \ldots\ldots \; m\sinh(m \cdot L)\cosh(m \cdot z)\right]_{z=0} \tag{I.30}$$

- as sinh 0 = 0 and cosh 0 = 1 then

$$q_z = k \cdot A \cdot m \cdot \theta_w \tanh(m \cdot L) \tag{I.31}$$

Fin efficiency and effectiveness

- The fin efficiency is defined as:

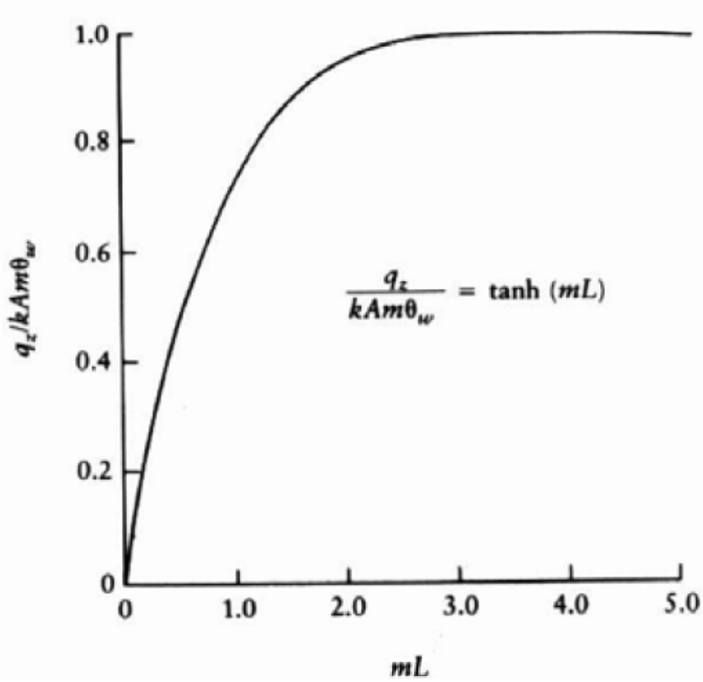
$$\eta_{fin} = \frac{\text{actual heat transferred from wall with fin attached}}{\text{heat that would be transferred if entire fin is at wall temperature}} \tag{I.32}$$

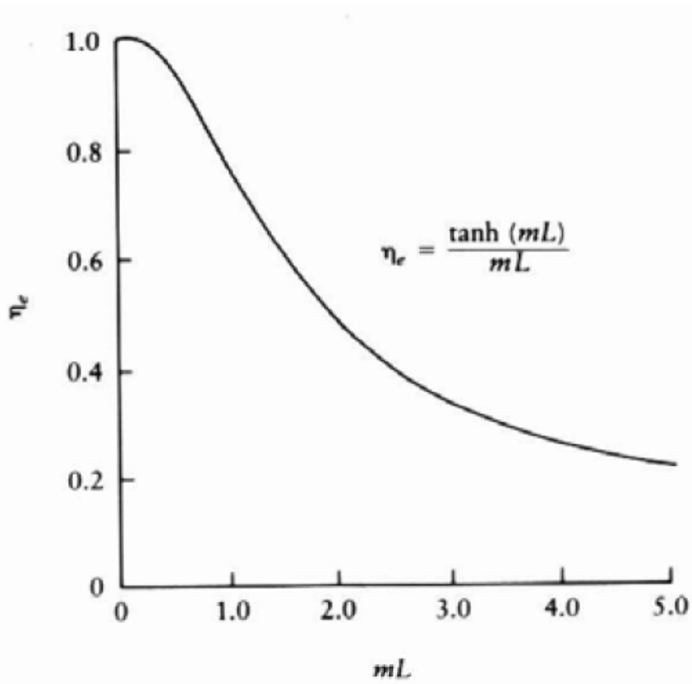
Thus

$$\eta_{fin} = \frac{\sqrt{k \cdot A \cdot \overline{h}_c \cdot P} \cdot (T_w - T_\infty)\tanh(m \cdot L)}{\overline{h}_c \cdot (P \cdot L)(T_w - T_\infty)} \tag{I.33}$$

$$\eta_{fin} = \frac{\tanh(m \cdot L)}{m \cdot L} \tag{I.34}$$

Fin efficiency and effectiveness





- Dimensionless graph of heat flow as a function of length for a uniform fin.

Efficiency of a fin.

Fin efficiency and effectiveness

- It can be seen that the heat rejected through the fin cannot be substantially increased past m x L=3 . Practically a fin length of over L=3/m will not improve the performance (m x L >3 is over designed).
- **The fin effectiveness is defined as:**

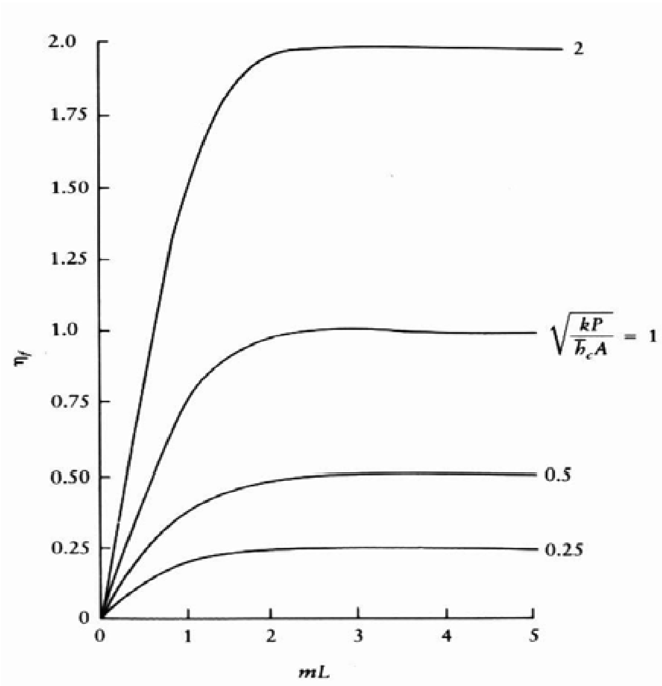
$$\varepsilon_{fin} = \frac{\text{heat flux from wall after adding fin}}{\text{heat flux from wall before adding fin}} \qquad (I.35)$$

- Thus

$$\varepsilon_{fin} = \sqrt{\frac{k \cdot P}{\bar{h}_c \cdot A}} \times \tanh(m \cdot L) \qquad (I.36)$$

- It can be seen that, fin will increase heat transfer rate when $\varepsilon_{fin}>1$ and the effectiveness increase when k is high and $\bar{h}_c$ is low. Thus, install fin may not increase the heat transfer rate if the value of $\bar{h}_c$ is large and the material k is low.

Fin efficiency and effectiveness

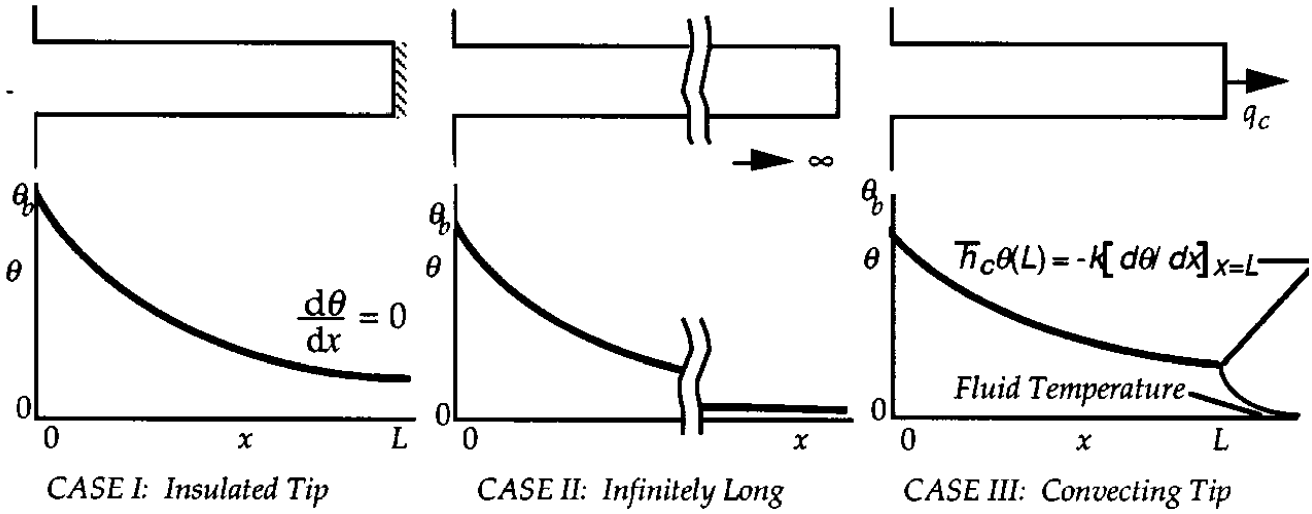


Effectiveness of a fin.

APPLICATION OF FINS

- **Fins** are widely used to enhance the heat transfer (usually convective, but it could also be radiative) from a surface. This is particularly true when the surface is in contact with a gas.
- Fins are used on air-cooled engines, electronic cooling forms, as well as for a number of other applications.
- Since the heat transfer coefficient tends to be low in gas convection, area is added in the form of fins to the surface to decrease the convective thermal resistance.
- The simplest fins to analyze, and which are usually found in practice, can be assumed to be one dimensional and constant in cross section. In simple terms, to be one-dimensional, the fins have to be long compared with a transverse dimension. Three cases are normally considered for analysis, and these are shown in Figure below. They are the insulated tip, the infinitely long fin, and the convecting tip fin.

APPLICATION OF FINS



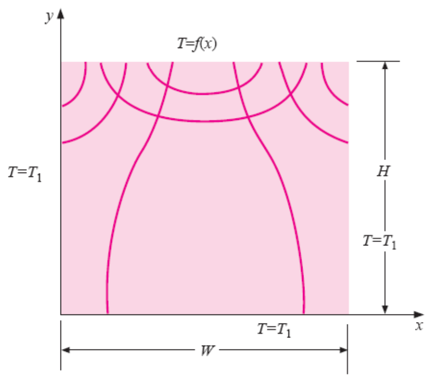
- Where $\theta_b = \theta_w$ = base or wall temperature.

2D HEAT TRANSFER

- The steady state with no heat generation, the Laplace equation

$$\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} = 0 \qquad \text{(J.1)}$$

- The objective of any heat-transfer analysis is usually to predict heat flow or the temperature that results from a certain heat flow
- Consider the rectangular plate shown in Figure. Three sides of the plate are maintained at the constant temperature *T1, and the upper side has some temperature distribution* impressed upon it. This distribution could be simply a **constant temperature** or something more complex, such as a **sine-wave distribution**



- Let the solution of equ.(J.1) is

$$T = XY$$
$$where X = X(x) \qquad (J.2)$$
$$Y = Y(y)$$

- Boundary conditions

T=T1 at y=0

T=T1 at x=0

T=T1 at x=W

$T = T_m \sin(\Pi x / W) + T_1$ at y=H (J.3)

T_m is the amplitude of sine wave

Substituting eqn(J.2) in (J.1)

$$-\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} \qquad (J.4)$$

- Observe that each side of Equation (J.4) is independent of the other because *x and y are* independent variables. This requires that each side be equal to some constant. Thus it may be two ordinary differential equations in terms of this constant,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} - \lambda^2 Y = 0 \qquad (J.5)$$

- Note that the form of the solution to Equations (J.5) will depend on the sign of $\lambda 2$

λ^2	solution	Using boundary condn. (J.3)
0	$X = C_1 + C_2 x$ $Y = C_3 + C_4 y$ $T = (C_1 + C_2 x)(C_3 + C_4 y)$	$C_1 C_3 = T_1$ $C_2 = 0$
< 0	$X = C_5 e^{-\lambda x} + C_6 e^{\lambda x}$ $Y = C_7 \cos \lambda y + C_8 \sin \lambda y$ $T = (C_5 e^{-\lambda x} + C_6 e^{\lambda x})(C_7 \cos \lambda y + C_8 \sin \lambda y)$	
> 0	$X = C_9 \cos \lambda x + C_{10} \sin \lambda x$ $Y = C_{11} e^{-\lambda y} + C_{12} e^{\lambda y}$ $T = (C_9 \cos \lambda x + C_{10} \sin \lambda x)(C_{11} e^{-\lambda y} + C_{12} e^{\lambda y})$	$C_{11} = -C_{12}$ $C_9 = 0$ $\sin \lambda W = 0$ $\lambda = \frac{n\Pi}{W}$

- Final solution

$$T = T_m \frac{\sinh(\Pi y / W)}{\sinh(\Pi H / W)} \sin(\Pi x / W) + T_1$$

- Boundary conditions

T=T1 at y=0

T=T1 at x=0

T=T1 at x=W

T=T2 at y=H

Temperature distribution:

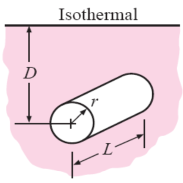
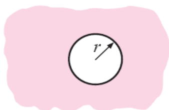
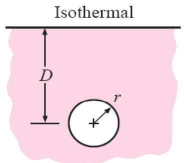
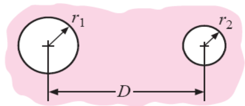
$$\frac{T - T_1}{T_2 - T_1} = \frac{2}{\Pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin(\Pi x / W) \frac{\sinh(\Pi y / W)}{\sinh(\Pi H / W)}$$

Shape factor

- In a two-dimensional system where only two temperature limits are involved, a conduction shape factor *S may be* define *as*

$$q = kS\Delta T_{overall}$$

$$S = shapefactor$$

Physical system	Schematic	Shape factor	Restrictions
Isothermal cylinder of radius r buried in semi-infinite medium having isothermal surface		$\frac{2\pi L}{\cosh^{-1}(D/r)}$ $\frac{2\pi L}{\ln(D/r)}$	$L \gg r$ $L \gg r$, $D > 3r$
Isothermal sphere of radius r buried in infinite medium		$4\pi r$	
Isothermal sphere of radius r buried in semi-infinite medium having isothermal surface $\Delta T = T_{surf} - T_{far\ field}$		$\frac{4\pi r}{1 - r/2D}$	
Conduction between two isothermal cylinders of length L buried in infinite medium		$\frac{2\pi L}{\cosh^{-1}\left(\frac{D^2 - r_1^2 - r_2^2}{2r_1r_2}\right)}$	$L \gg r$ $L \gg D$